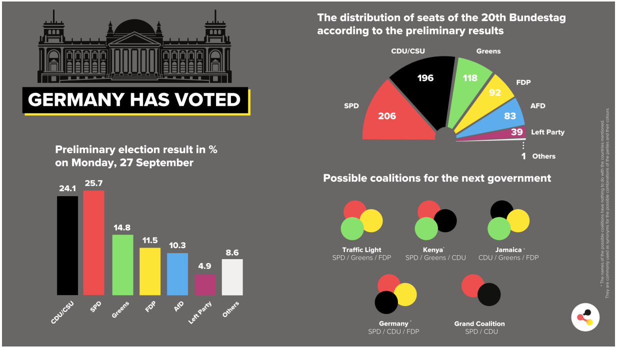
Topic 3: Coalitional Games



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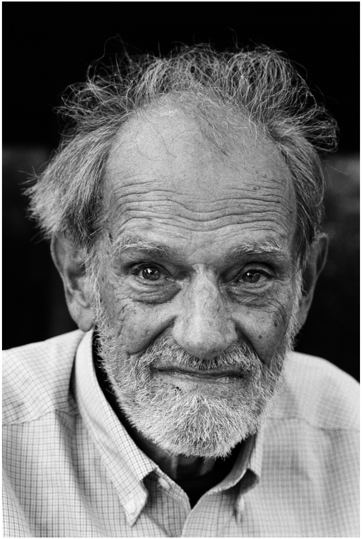
Outcomes & Objectives

- Be proficient in solving coalitional games
  - Model player's rationality in forming coalitions via defining a value of a given coalition.
  - Identify some useful subclasses of games which produces some special coalitions.
  - Develop a solution concept called *Shapley value* to distribute a coalition's value in a fair manner.
  - Develop a solution concept called *core* that identifies a stable coalition structure in the game.

Lloyd Shapley



Shapley was the greatest game theorist of all time.

– Robert Aumann

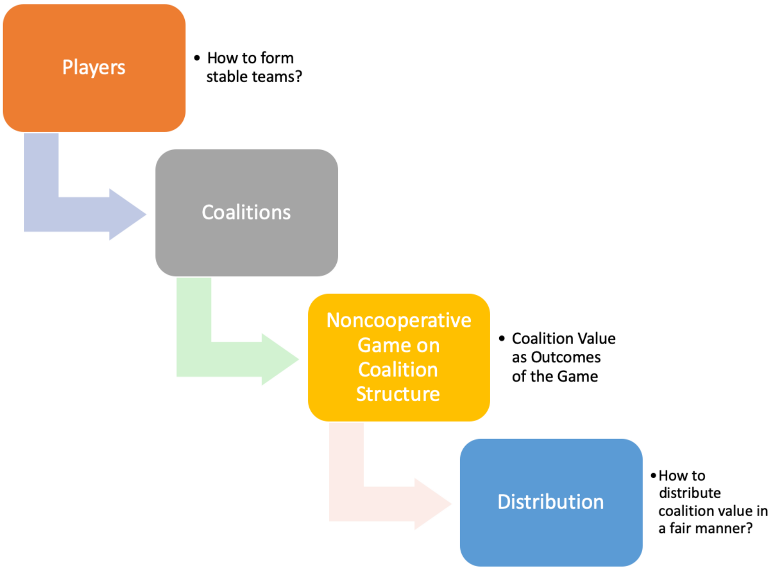
Applications of Coalitional Games

- **Political Coalitions:** Parties form coalitions if the elections did not result in one party with a majority votes. Coalitional governments resolve such concerns. However, the question is which coalitions form stable governments.

- **Cost Sharing for Network Design:** Users benefit from being connected to a server. So they have to build up a broadcast tree. However, it costs to maintain the server/network and the question is how to share the costs.

- **Queue Management:** Multiple users want to route traffic through a switch, which has a flow dependent delay (cost). The queueing delay cost has to be shared among the users.

Coalitional Game: An Overview



Coalitions and Transferable Utilities

Definition

Given a set of players $\mathcal{N} = \{1, \cdots, N\}$, a ***coalition*** is a subset of $\mathcal{N}$. Furthermore, a ***grand coalition*** is the set of all players $\mathcal{N}$.

Definition

A ***characteristic function game*** Γ is a pair $(\mathcal{N}, v)$, where $\mathcal{N}$ is the set of players, and $v : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ is a ***characterisic function***, which assigns each coalition $\mathcal{C} \subseteq \mathcal{N}$, some real value $v(\mathcal{C})$.

Definition

A characteristic function game $\Gamma = (\mathcal{N}, v)$ is a ***transferable utility game***, if the value of any coalition $v(\mathcal{C})$ can be distributed amongst the members in $\mathcal{C}$ in any way that the members of $\mathcal{C}$ choose.

Standard Assumptions:

- The value of a empty coalition is 0.
- $v(\mathcal{C}) \geq 0$, for any $\mathcal{C} \subseteq \mathcal{N}$.

Example

A fictional country X has a 101-member parliament, where each representative belongs to one of the three parties:

- Liberal (L): 40 representatives
- Moderate (M): 31 representatives
- Conservative (C): 30 representatives

The parliament needs to decide how to allocate \$1bn of discretionary spending, and each party has its own preferred way of using this money. The decision is made by a simple **majority vote**, and we assume that all representatives vote along the party lines.

Parties can form **coalitions**; a coalition has value \$1bn if it can win the budget vote no matter what the other parties do, and value 0 otherwise.

This situation can be modeled as a three-player characteristic function game, where the set of players is $\mathcal{N} = L, M, C$ and the characteristic function is given by

$$v(\mathcal{C}) = \begin{cases} 0, & \text{if } |\mathcal{C}| \leq 1, \\ 10^9, & \text{otherwise.} \end{cases}$$

Coalition Structure

Definition

Given a characteristic function game $\Gamma = (\mathcal{N}, v)$, a ***coalition structure*** $\boldsymbol{\mathcal{C}}$ is a partition of $\mathcal{N}$. In other words, $\boldsymbol{\mathcal{C}}$ is a collection of non-empty subsets $\{\mathcal{C}_1, ..., \mathcal{C}_K\}$ such that

- $\bigcup_{k \in \{1,...,K\}} \mathcal{C}_k = \mathcal{N}$, and
- $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$, for any $i, j \in \{1, ..., K\}$ such that $i \neq j$.

Definition

A vector $\boldsymbol{u} = \{u_1, \cdots, u_N\} \in \mathbb{R}^N$ is the ***utility profile*** for a coalition structure $\boldsymbol{\mathcal{C}} = \{\mathcal{C}_1, \cdots, \mathcal{C}_K\}$ over $\mathcal{N}$ if

- ***Non-Negativity:*** $u_i \geq 0$ for all $i \in \mathcal{N}$, and
- ***Feasibility:*** $\sum_{i \in \mathcal{C}_k} u_i \leq v(\mathcal{C}_k)$ for any $k \in \{1, \cdots, K\}$.

Outcome, Efficiency and Social Welfare

Definition

The ***outcome*** of a game Γ is a pair $(\boldsymbol{\mathcal{C}}, \boldsymbol{u})$.

Definition

An outcome $(\boldsymbol{\mathcal{C}}, \boldsymbol{u})$ is ***efficient***, if all the utilities are distributed amongst the coalition members, i.e.

$$\sum_{i \in \mathcal{C}_k} u_i = v(\mathcal{C}_k), \text{ for all } k = 1, \cdots, K.$$

Definition

The social welfare of a coalition structure $\mathcal{C}$ is

$$v(\boldsymbol{\mathcal{C}}) = \sum_{k=1}^{K} v(\mathcal{C}_k)$$

Individual Rationality and Imputation

Definition

A player i is said to be ***individually rational*** in an outcome $(\mathcal{C}, \boldsymbol{u})$, if

$$u_i \geq v(\{i\}),$$

where $v(\{i\})$ is the value of the coalition $\{i\}$, which only contains the i^{th} player.

Definition

A outcome $(\mathcal{C}, \boldsymbol{u})$ is said to be an ***imputation***, if it is efficient, and if every player is individually rational within itself.

- Each player weakly prefers being in the coalition structure, than being on his/her own.
- Group deviations $\Rightarrow$ Stability of Coalitions (covered later)

Monotone Games

Definition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is said to be ***monotone*** if it satisfies $v(\mathcal{C}) \leq v(\mathcal{D})$, for every pair of coalitions $\mathcal{C}, \mathcal{D} \subseteq \mathcal{N}$, such that $\mathcal{C} \subseteq \mathcal{D}$.

- Most games are monotone!
- However, non-monotonicity may arise because
  - some players intensely dislike each other, or
  - communication costs increase nonlinearly with coalition size.

Example: Three commuters can share a taxi. Individual journey costs: $P_1 : 6$, $P_2 : 12$, $P_3 : 42$. Then, the following characteristic function results in a monotone game:

$$v_1(\mathcal{C}) = \begin{cases} 6 & \text{if } \mathcal{C} = \{1\} \\ 12 & \text{if } \mathcal{C} = \{2\} \\ 42 & \text{if } \mathcal{C} = \{3\} \\ 12 & \text{if } \mathcal{C} = \{1,2\} \\ 42 & \text{if } \mathcal{C} = \{1,3\} \\ 42 & \text{if } \mathcal{C} = \{2,3\} \\ 42 & \text{if } \mathcal{C} = \{1,2,3\}. \end{cases}$$

Superadditive Games

Definition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is said to be ***superadditive*** if it satisfies $v(\mathcal{C} \cup \mathcal{D}) \geq v(\mathcal{C}) + v(\mathcal{D})$, for every pair of disjoint coalitions $\mathcal{C}, \mathcal{D} \subseteq \mathcal{N}$.

Proposition

If a superadditive game $\Gamma = \{\mathcal{N}, v\}$ has a non-negative characteristic function v, then Γ is monotone.

Proof: For any pair of coalitions $\mathcal{C} \subseteq \mathcal{D}$, we have

$$v(\mathcal{C}) \leq v(\mathcal{D}) - v(\mathcal{D} - \mathcal{C}) \leq v(\mathcal{D}).$$

$\square$

- Monotonicity $\not\Rightarrow$ superadditivity. (Example: $v(\mathcal{C}) = \log |\mathcal{C}|$.)
- Always profitable for two groups to join forces $\Rightarrow$ Grand Coalition.
- *Anti-trust* or *anti-monopoly* laws $\Rightarrow$ Non-superadditive games.

Superadditive Games: Example

Consider the same taxi example:

- Three commuters can share a taxi. Individual journey costs: $P_1 : 6$, $P_2 : 12$, $P_3 : 42$.
- Then, $v_1(\mathcal{C})$ is not superadditive.
- However, the following characteristic function results in a superadditive game:

$$v_2(\mathcal{C}) = \begin{cases} 6 & \text{if } \mathcal{C} = \{1\} \\ 12 & \text{if } \mathcal{C} = \{2\} \\ 42 & \text{if } \mathcal{C} = \{3\} \\ 18 & \text{if } \mathcal{C} = \{1,2\} \\ 48 & \text{if } \mathcal{C} = \{1,3\} \\ 55 & \text{if } \mathcal{C} = \{2,3\} \\ 80 & \text{if } \mathcal{C} = \{1,2,3\}. \end{cases}$$

Convex Games

Definition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is said to be ***convex*** if the characteristic function v is supermodular, i.e., it satisfies $v(\mathcal{C} \cup \mathcal{D}) + v(\mathcal{C} \cap \mathcal{D}) \geq v(\mathcal{C}) + v(\mathcal{D})$ for every pair of coalitions $\mathcal{C}, \mathcal{D} \subseteq \mathcal{N}$.

Proposition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is convex, if and only if, for every pair of coalitions $\mathcal{C}, \mathcal{D}$ such that $\mathcal{C} \subset \mathcal{D}$, and for every player $i \in \mathcal{N} - \mathcal{D}$, we have

$$v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leq v(\mathcal{D} \cup \{i\}) - v(\mathcal{D})$$

- Players become more useful if they join bigger coalitions.
- Convexity $\Rightarrow$ Superadditivity.
- However, the converse may not be true!

  3-player majority game: Consider a game $\Gamma = (\mathcal{N}, v)$, where $\mathcal{N} = \{1, 2, 3\}$, and $v(\mathcal{C}) = 1$ if $|\mathcal{C}| \geq 2$, and $v(\mathcal{C}) = 0$ otherwise. This game is superadditive. On the other hand, for $\mathcal{C} = \{1, 2\}$ and $\mathcal{D} = \{2, 3\}$, we have $v(\mathcal{C}) = v(\mathcal{D}) = 1$, $v(\mathcal{C} \cup \mathcal{D}) = 1$, $v(\mathcal{C} \cap \mathcal{D}) = 0$.

Simple Games

Definition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is said to be ***simple*** if it is monotone and its characteristic function only takes values 0 and 1, i.e. $v(\mathcal{C}) \in \{0, 1\}$, for any $\mathcal{C} \subseteq \mathcal{N}$.

- $v(\mathcal{C}) = 1 \;\Rightarrow$ Winning Coalition.
- $v(\mathcal{C}) = 0 \;\Rightarrow$ Loosing Coalition.

Claim

A simple game $\Gamma = \{\mathcal{N}, v\}$ is superadditive, only if the complement of every winning coalition looses.

Solution Concepts

Outcomes can be evaluated based on two sets of criteria:

- **Fair Distribution:** How well each agent's payoff reflects his/her contribution?
  - Shapley Value
  - Banzhaf Index
- **Coalition Stability:** What are the incentives for the agents to stay in the coalition structure?
  - Stable Set
  - Core
  - Nucleolus
  - Bargaining Set

Fair Distribution: Shapley's Axioms

Let u_i^Γ denote the allocation (utility) to the i^{th} player in a game $\Gamma = \{\mathcal{N}, v\}$. Then, we desire the following four properties:

- **Efficiency:** Distribute the value of grand coalition to all agents, i.e.

$$\sum_{i \in \mathcal{N}} u_i^\Gamma = v(\mathcal{N}).$$

- **Dummy Player:** If a player i does not contribute to any coalition in Γ, then

$$u_i^\Gamma = 0.$$

- **Symmetry:** If two players i and j contribute equally to each coalition in Γ, then

$$u_i^\Gamma = u_j^\Gamma.$$

- **Additivity:** If the same set of players are involved in two coalitional games $\Gamma_1 = (\mathcal{N}, v_1)$ and $\Gamma_2 = (\mathcal{N}, v_2)$, if we define $\Gamma = \Gamma_1 + \Gamma_1 = (\mathcal{N}, v_1 + v_2)$, then for every player i, we have

$$u_i^\Gamma = u_i^{\Gamma_1} + u_i^{\Gamma_2}.$$

Finding a Fair Distribution...

Assume we have a superadditive game, which results in a grand coalition!

- Agent's allocation is proportional to his/her contribution in $v(\mathcal{N})$.
- Idea: As each agent joins to form the grand coalition, compute how much the value of the coalition increases, i.e., allocate $u_i = v(\mathcal{N}) - v(\mathcal{N} - \{i\})$ to player i.

This contribution is evaluated when the player is the last inclusion in $\mathcal{N}$.

But, what about players who joined the coalition before the last player?

Let $\Pi_{\mathcal{N}}$ denote the set of all permutations of $\mathcal{N}$, i.e., one-to-one mappings from $\mathcal{N}$ to itself. Given a permutation $\pi \in \Pi_{\mathcal{N}}$, we denote by $S_\pi(i)$ the set of all predecessors of i in π, i.e., we set

$$S_\pi(i) = \{ \ j \in \mathcal{N} \mid \pi(j) < \pi(i) \ \}.$$

Example: If $\mathcal{N} = \{1, 2, 3\}$, we have

$$\Pi_{\mathcal{N}} = \{\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \{3,2,1\}\}.$$

Then, if $\pi = \{2, 1, 3\}$, we have

$$S_\pi(2) = \emptyset \quad S_\pi(1) = \{2\} \quad S_\pi(3) = \{1, 2\}$$

Shapley Value

Definition

The ***marginal contribution*** of an agent i with respect to a permutation π in a game $\Gamma = (\mathcal{N}, v)$ is given by

$$\Delta_\pi^\Gamma(i) = v\left[S_\pi(i) \cup \{i\}\right] - v\left[S_\pi(i)\right].$$

Definition

Given a characteristic function game $\Gamma = (\mathcal{N}, v)$ with $|\mathcal{N}| = N$, the ***Shapley value*** of an agent $i \in \mathcal{N}$ is given by

$$u_i(\Gamma) = \frac{1}{N!} \sum_{\pi \in \Pi_{\mathcal{N}}} \Delta_\pi^\Gamma(i).$$

Theorem

Shapley's axioms *uniquely* characterize Shapley value. In other words, Shapley value is the only fair distribution scheme that satisfies all the Shapley's axioms.

Shapley Value: Example

Consider the same ridesharing example, as stated earlier.

- Three commuters can share a taxi.
- Individual journey costs: $P_1 : 6$, $P_2 : 12$, $P_3 : 42$.
- The characteristic function is

$$v_1(\mathcal{C}) = \begin{cases} 6 & \text{if } \mathcal{C} = \{1\} \\ 12 & \text{if } \mathcal{C} = \{2\} \\ 42 & \text{if } \mathcal{C} = \{3\} \\ 12 & \text{if } \mathcal{C} = \{1,2\} \\ 42 & \text{if } \mathcal{C} = \{1,3\} \\ 42 & \text{if } \mathcal{C} = \{2,3\} \\ 42 & \text{if } \mathcal{C} = \{1,2,3\}. \end{cases}$$

Permutation set $\Pi_{\mathcal{N}} = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$, where

$$\pi_1 = \{1,2,3\}, \quad \pi_2 = \{1,3,2\}, \quad \pi_3 = \{2,1,3\},$$
$$\pi_4 = \{2,3,1\}, \quad \pi_5 = \{3,1,2\}, \quad \pi_6 = \{3,2,1\}.$$

Shapley Value: Example (cont...)

Given

$$\pi_1 = \{1,2,3\}, \quad \pi_2 = \{1,3,2\},$$
$$\pi_3 = \{2,1,3\}, \quad \pi_4 = \{2,3,1\}, \qquad \text{and} \qquad v_1(\mathcal{C}) = \begin{cases} 6 & \text{if } \mathcal{C} = \{1\} \\ 12 & \text{if } \mathcal{C} = \{2\} \\ 42 & \text{if } \mathcal{C} = \{3\} \\ 12 & \text{if } \mathcal{C} = \{1,2\} \\ 42 & \text{if } \mathcal{C} = \{1,3\} \\ 42 & \text{if } \mathcal{C} = \{2,3\} \\ 42 & \text{if } \mathcal{C} = \{1,2,3\}. \end{cases}$$
$$\pi_5 = \{3,1,2\}, \quad \pi_6 = \{3,2,1\},$$

Marginal contributions:

- π_1: $\Delta_1^\Gamma(1) = 6$, $\Delta_1^\Gamma(2) = 6$, $\Delta_1^\Gamma(3) = 30$
- π_2: $\Delta_2^\Gamma(1) = 6$, $\Delta_2^\Gamma(2) = 0$, $\Delta_2^\Gamma(3) = 36$
- π_3: $\Delta_3^\Gamma(1) = 0$, $\Delta_3^\Gamma(2) = 12$, $\Delta_3^\Gamma(3) = 30$
- π_4: $\Delta_4^\Gamma(1) = 0$, $\Delta_4^\Gamma(2) = 12$, $\Delta_4^\Gamma(3) = 30$
- π_5: $\Delta_5^\Gamma(1) = 0$, $\Delta_5^\Gamma(2) = 0$, $\Delta_5^\Gamma(3) = 42$
- π_6: $\Delta_6^\Gamma(1) = 0$, $\Delta_6^\Gamma(2) = 0$, $\Delta_6^\Gamma(3) = 42$

Shapley value:

- $u_1(\Gamma) = \dfrac{1}{6} \displaystyle\sum_{i=1}^{6} \Delta_i^\Gamma(1) = 2$
- $u_2(\Gamma) = \dfrac{1}{6} \displaystyle\sum_{i=1}^{6} \Delta_i^\Gamma(2) = 5$
- $u_3(\Gamma) = \dfrac{1}{6} \displaystyle\sum_{i=1}^{6} \Delta_i^\Gamma(3) = 35$

Stability of Coalitions: Core

- Consider a characteristic function game $\Gamma = \{\mathcal{N}, v\}$ with an outcome $(\boldsymbol{\mathcal{C}}, \boldsymbol{u})$.
- Let $u(\mathcal{C})$ denote the total payoff of a coalition $\mathcal{C}$ under $\boldsymbol{u}$.
- Given a coalition $\mathcal{C}$, if $u(\mathcal{C}) < v(\mathcal{C})$, some agents can abandon $\mathcal{C}$ and form their own coalition.

Definition

A utility profile $\boldsymbol{u}$ is ***stable*** through a coalition $\mathcal{C}$ if

$$\sum_{i \in \mathcal{C}} u_i \geq v(\mathcal{C}).$$

Definition

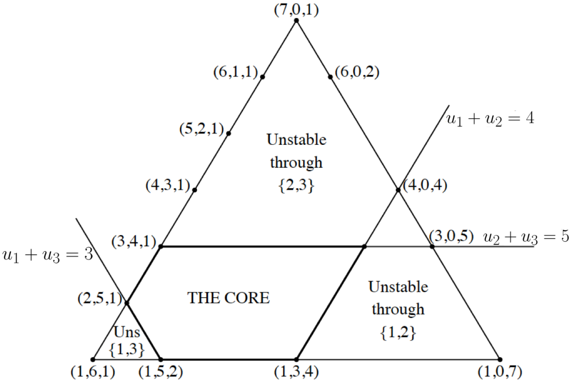
Core is defined as the set of all stable utility profiles, which is denoted as

$$\mathbb{C} = \left\{ \boldsymbol{u} \in \mathbb{R}_+^N \;\middle|\; \sum_{i \in \mathcal{C}} u_i \geq v(\mathcal{C}), \text{ for all } \mathcal{C} \subset \mathcal{N} \right\}.$$

Core: An Example

Consider a characteristic function game $\Gamma = \{\mathcal{N}, v\}$, where $\mathcal{N} = \{1, 2, 3\}$ and

$$v(\mathcal{C}) = \begin{cases} 1 & \text{if } \mathcal{C} = \{1\} \\ 0 & \text{if } \mathcal{C} = \{2\} \\ 1 & \text{if } \mathcal{C} = \{3\} \\ 4 & \text{if } \mathcal{C} = \{1,2\} \\ 3 & \text{if } \mathcal{C} = \{1,3\} \\ 5 & \text{if } \mathcal{C} = \{2,3\} \\ 8 & \text{if } \mathcal{C} = \{1,2,3\}. \end{cases}$$



- Then, the utility profiles are those such that $u_1 + u_2 + u_3 = 8$ such that $u_1 \geq 1$, $u_2 \geq 0$ and $u_3 \geq 1$.
- This is a hyperplane with vertices $(7, 0, 1)$, $(1, 0, 7)$, and $(1, 6, 1)$.

Is core always non-empty?

Core in Convex and Simple Games

Theorem

Any convex game $\Gamma = (\mathcal{N}, v)$ has a non-empty core.

Definition

In a characteristic function game $\Gamma = (\mathcal{N}, v)$, a player i is called a ***veto player***, if $v(\mathcal{C}) = 0$ for all $\mathcal{C} \subseteq \mathcal{N} - \{i\}$.

Theorem

A simple game $\Gamma = (\mathcal{N}, v)$ has a non-empty core, if and only if there is a veto player in $\mathcal{N}$. Moreover, a utility profile $\boldsymbol{u}$ is in the core of Γ if and only if $u_i = 0$ for every player i, who is not a veto player in Γ.

Core and Superadditive Covers

Definition

$\Gamma^* = (\mathcal{N}, v^*)$ is called a ***superadditive cover*** of $\Gamma = (\mathcal{N}, v)$ if, for every coalition $\mathcal{C} \subseteq \mathcal{N}$,

$$v^*(\mathcal{C}) = \max_{\mathcal{P}_\mathcal{C}} \sum_{\mathcal{C}_i \in \mathcal{P}_\mathcal{C}} v(\mathcal{C}_i),$$

where $\mathcal{P}_\mathcal{C}$ denotes a partition of the coalition $\mathcal{C}$.

Consider $\Gamma = (\mathcal{N}, v)$: $\mathcal{N} = \{1, 2, 3\}$ and

$$v(\mathcal{C}) = \begin{cases} 5 & \text{if } \mathcal{C} = \{1\} \\ 0 & \text{if } \mathcal{C} = \{2\} \\ 0 & \text{if } \mathcal{C} = \{3\} \\ 1 & \text{if } \mathcal{C} = \{1, 2\} \\ 1 & \text{if } \mathcal{C} = \{1, 3\} \\ 1 & \text{if } \mathcal{C} = \{2, 3\} \\ 1 & \text{if } \mathcal{C} = \{1, 2, 3\}. \end{cases}$$

Its superadditive cover $\Gamma^* = (\mathcal{N}, v^*)$ is

$$v^*(\mathcal{C}) = \begin{cases} 5 & \text{if } \mathcal{C} = \{1\} \\ 0 & \text{if } \mathcal{C} = \{2\} \\ 0 & \text{if } \mathcal{C} = \{3\} \\ 5 & \text{if } \mathcal{C} = \{1, 2\} \\ 5 & \text{if } \mathcal{C} = \{1, 3\} \\ 1 & \text{if } \mathcal{C} = \{2, 3\} \\ 6 & \text{if } \mathcal{C} = \{1, 2, 3\}. \end{cases}$$

Theorem

A characteristic function game $\Gamma = (\mathcal{N}, v)$ has a non-empty core if and only if its superadditive cover $\Gamma^* = (\mathcal{N}, v^*)$ has a non-empty core.

Summary

- **Characteristic function game:** How to model players' rationality in coalitional games?
- **Subclasses:** Are there any special games that result in some specific coalitions?
- **Shapley value:** How to distribute a coalition's value in a fair manner amongst its members?
- **Core:** What is a stable coalition?