Topic 1: Decision Theory

Outcomes & Objectives

- Master the ability to study/model and analyze agent choices (lotteries) using *expected utility theory*.
  - Demonstrate why people do not maximize expected rewards.
  - Characterize agent's choice preferences using utilities, and outcome beliefs using subjective probabilities.
  - Develop an intuitive axiomatic framework in which agent picks choices to maximize his/her expected utility.
- Illustrate the limitations of EUT and formulate models to better accomodate various deviating behaviors.
  - Relate axiom violations to other well-known normative models.
  - Identify certain deviations from experiments and associate them with descriptive models.
- Devise and become proficient in a decision model based on *domination*, when agents cannot evaluate beliefs.

Philosophy of Decision Theory

- **Agent:** One decision maker (or a team of multiple decision makers working in tandem) in a given system.
- **Agent Rationality:** The philosophy (principle) used by agents to make decisions.

How does an agent make decisions under uncertainty?

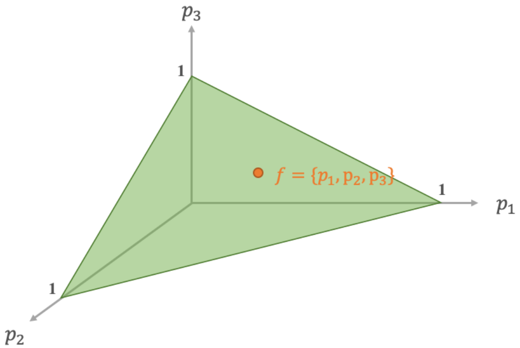
Can we model agent rationality mathematically?

Two fundamental approaches:

- **Normative Models:** Identification of optimal decision outcomes – Prescriptive in nature
- **Descriptive Models:** Describe observed behaviors using consistent rules/models.

Modeling Choice Uncertainty

- $\mathcal{X}$: (Discrete) Set of all possible prizes
- $\Delta(\mathcal{X})$: Set of all possible randomizations over choices, called a probability *simplex*



However, these choice probabilities are conditional to the information available at the agent.

Modeling Choice Uncertainty (cont...)

- Say, you have two route choices: A and B.
- Normally, decide A or B based on preference evaluation.

What if there was an unexpected accident in route A?

- Outcomes rely on the state of the choice experiment.
- Ω: Set of all possible states.

Definition

A lottery is any probability distribution $f : \Omega \to \Delta(\mathcal{X})$ that specifies a non-negative number $f(x|t)$ for every prize $x \in \mathcal{X}$ and every state $t \in \Omega$ such that $\sum_{x \in \mathcal{X}} f(x|t) = 1$ holds true for every state $t \in \Omega$.

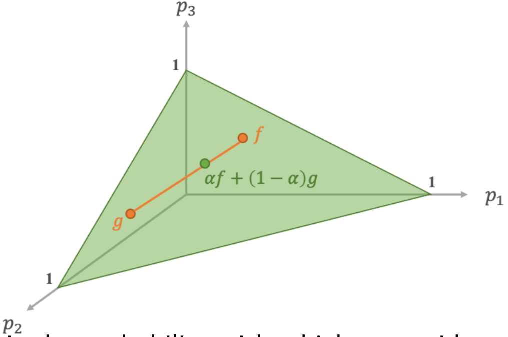
- $\mathcal{L}$: Set of all such lotteries in the choice experiment.

Randomization of Lotteries

Given two lotteries $f, g \in \mathcal{L}$, and a number $\alpha \in [0,1]$, the lottery $\alpha f + (1-\alpha)g$ denotes a lottery in $\mathcal{L}$ such that

$$(\alpha f + (1-\alpha)g)(x|t) = \alpha f(x|t) + (1-\alpha)g(x|t)$$

for all $x \in \mathcal{X}$ and $t \in \Omega$.



Example: α is the probability with which an accident can take place in route A.

Preference Relations

Consider any two lotteries $f, g \in \mathcal{L}$ (finite, countably infinite, or uncountable) be given. Assume that an event $E \in \mathcal{E}$ has been observed that reveals state information.

- **Binary Relation:** $f \ \succ_E \ g$.

  Example: f *is greater than* g, f *is more tasty than* g

- **Negation:** $\neg(f \succ_E g)$.

  Example: f *is not greater than* g, f *is not as tasty as* g

- **Indifference:** $f \sim_E g$.

  Example: f *is equal to* g, f *is similar (incomparable) in taste to* g

- **Weak Relation:** $f \succsim_E g$.

  Example: f *is greater than or equal to* g, f *is at least as tasty as* g

St. Petersburg Paradox

(Invoked in Blaise Pascal's Wager, published in *Pensèes* in 1670)

Consider the following choice experiment...

T = 1	**T = 2**	**T = 3**	**T = 4**
Outcome:	**Outcome:**	**Outcome:**	**Outcome:**
Win 2, if heads in T = 1	Win 4, if heads in T = 1, 2	Win 8, if heads in T = 1, 2, 3	Win 16, if heads in T = 1, 2, 3, 4
Win 0, otherwise.	Win 0, otherwise.	Win 0, otherwise.	Win 0, otherwise.

Final outcome: Accumulate all the rewards obtained over all time instances of the experiment.

Can you formally state this experiment as a set of lotteries?

If you were to choose the length of play beforehand, how long would you play this game?

St. Petersburg Paradox (cont...)

(Invoked in Blaise Pascal's Wager, published in *Pensèes* in 1670)

T = 1
Outcome:
Win 2, if heads in T = 1
Win 0, otherwise.

T = 2
Outcome:
Win 4, if heads in T = 1, 2
Win 0, otherwise.

T = 3
Outcome:
Win 8, if heads in T = 1, 2, 3
Win 0, otherwise.

T = 4
Outcome:
Win 16, if heads in T = 1, 2, 3, 4
Win 0, otherwise.

$$\begin{aligned} \text{Expected reward} &= 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + \cdots \\ &= 1 + 1 + \cdots = \infty. \end{aligned}$$

Prescription: Play over an infinite time-horizon!

But, any sane person chooses a finite time-horizon!

In other words, we do not maximize expected rewards!

St. Petersburg Paradox (cont...)

(Daniel Bernoulli in *Commentaries of the Imperial Academy of Science of Saint Petersburg* in 1738)

T = 1	T = 2	T = 3	T = 4
Outcome:	**Outcome:**	**Outcome:**	**Outcome:**
Win 2, if heads in T = 1	Win 4, if heads in T = 1, 2	Win 8, if heads in T = 1, 2, 3	Win 16, if heads in T = 1, 2, 3, 4
Win 0, otherwise.	Win 0, otherwise.	Win 0, otherwise.	Win 0, otherwise.

Decreasing Marginal Utilities:

$$\begin{aligned}
\text{Expected utility} &= \ln 2 \times \frac{1}{2} + \ln 4 \times \frac{1}{4} + \cdots \\
&= \left[\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{k}{2^k} + \cdots\right] \ln 2 \\
&< \infty.
\end{aligned}$$

However, logarithmic distortion of rewards does not characterize diverse choice preferences across different agents.

Ordinal Utility

Definition

Ordinal utility is any deterministic function $u : \mathcal{X} \times \Omega \to \mathbb{R}$ such that, for any $E \subseteq \Omega$,

$$x_1 \succsim_E x_2 \iff u(x_1, t) \geq u(x_2, t) \text{ for all } x_1, x_2 \in \mathcal{X}, t \in E.$$

Example: Consider an agent who is presented with four choice outcomes, $\mathcal{X} = \{a, b, c, d\}$ and Ω contains only one state. Let the agent's preference ordering be $a \succ b \succ c \succ d$.

Then, how can we assign real numbers to outcomes so as to reflect the above preference ordering?

Assignments: Uncountably infinite possibilities...

- $a : 4, b : 3, c : 2, d : 1, \qquad a : 100, b : 50, c : 10, d : 0$

Such assignments do not capture the degree of agent's preferences.

Does such utility functions exist in all choice experiments?

Expected Utility

Definition

Given any conditional distribution p, utility function u, lottery $f \in \mathcal{L}$ and any event $E \in \mathcal{E}$, the expected utility of the prize determined by f is given by

$$\mathbb{E}_p(u(f)|E) = \sum_{t \in E} p(t|E) \sum_{x \in \mathcal{X}} u(x,t) f(x|t).$$

Expected Utility: Example

Suppose a commuter has two route choices A and B. Route A is a local road with utility $u_A = 0.3$. Route B is a highway route, which has a utility

$$u_B = \begin{cases} 1, & \text{if Route } A \text{ is normal} \\ 0.5, & \text{if Route } A \text{ is under construction} \end{cases}$$

if there is no accident, and

$$u_B = \begin{cases} 0.25, & \text{if Route } A \text{ is normal} \\ 0.1, & \text{if Route } A \text{ is under construction} \end{cases}$$

if there is an accident. Let Route B be under construction with probability 0.4. If the commuter takes Route B with probability 0.75, then the expected utilities are given by

$$\begin{aligned} EU_{NoAccident} &= 0.6 \times (1 \times 0.75 + 0.3 \times 0.25) + 0.4 \times (0.5 \times 0.75 + 0.3 \times 0.25) \\ &= 0.675 \end{aligned}$$

$$\begin{aligned} EU_{Accident} &= 0.6 \times (0.25 \times 0.75 + 0.3 \times 0.25) + 0.4 \times (0.1 \times 0.75 + 0.3 \times 0.25) \\ &= 0.2175 \end{aligned}$$

Axioms of Decision Theory

Basic properties that a rational agent's preferences may satisfy:

1. **Completeness:** Either $f \succ_E g$, or $g \succ_E f$. or $f \sim_E g$

2. **Transitivity:** If $f \succsim_E g$ and $g \succsim_E h$, then $f \succsim_E h$.

3. **Relevance:** If $f(\cdot|t) = g(\cdot|t)$ for all $t \in E$, then $f \sim_E g$.

4. **Monotonicity:** If $f \succsim_E g$ and $0 \leq \beta \leq \alpha \leq 1$, then

$$\alpha f + (1-\alpha)g \ \succsim_E \ \beta f + (1-\beta)g.$$

5. **Continuity:** If $f \succsim_E g$ and $g \succsim_E h$, then there exists $\alpha_g \in [0,1]$ such that

$$g \ \sim_E \ \alpha_g f + (1-\alpha_g)h.$$

Axioms of Decision Theory (cont...)

6. **Objective Substitution:** If $f_1 \succsim_E g_1$ and $f_2 \succsim_E g_2$ and $\alpha \in [0,1]$, then
$$\alpha f_1 + (1-\alpha) f_2 \ \succsim_E \ \alpha g_1 + (1-\alpha) g_2.$$
7. **Subjective Substitution:** If $f \succsim_{E_1} g$ and $f \succsim_{E_2} g$ and $E_1 \cap E_2 = \emptyset$, then $f \ \succsim_{E_1 \cup E_2} \ g$.
8. **Interest:** For every $t \in \Omega$, there exists at least one pair of prizes $x_1, x_2 \in \mathcal{X}$ such that $x_1 \succ_{\{t\}} x_2$.
9. **State Neutrality:** For any two states $s, t \in \Omega$, if $f(\cdot|s) = f(\cdot|t)$, $g(\cdot|s) = g(\cdot|t)$ and $f \succsim_{\{s\}} g$, then $f \succsim_{\{t\}} g$.

Expected Utility Maximization (EUM)

Credit: Von Neumann and Morgenstern, 1947

Theorem 1

Axioms 1-8 are jointly satisfied if and only if there exists a utility function $u : \mathcal{X} \times \Omega \to [0, 1]$ and a conditional probability function $p : \mathcal{E} \to \Delta(\Omega)$ such that

- $f \succsim_E g$ if and only if $\mathbb{E}_p(u(f)|E) \geq \mathbb{E}_p(u(g)|E)$ for all $f, g \in \mathcal{L}$ and for all $E \in \mathcal{E}$.

- If there are more than two lotteries, by transitivity, the most preferred lottery also has the largest expected utility!

State-Independent Utility Maximization

Credit: Von Neumann and Morgenstern, 1947

State-Independent Utility: $u(x,t) = U(x)$, for all t, x. (inspired from *State Neutrality* axiom)

Corollary 1

Axioms 1-9 are satisfied if and only if there exists a state-independent utility function $u : \mathcal{X} \times \Omega \to [0,1]$ and a conditional probability function $p : \mathcal{E} \to \Delta(\Omega)$ such that

- $f \succsim_E g$ if and only if $\mathbb{E}_p(u(f)|E) \geq \mathbb{E}_p(u(g)|E)$ for all $f, g \in \mathcal{L}$ and for all $E \in \mathcal{E}$.

Given that utility functions exist under Axioms 1-9, how can we construct[1] them from agents' revelations?

[1] This is beyond the scope of this course. However, interested students may refer to Revealed Preference Theory and Afriat's Theorem.

Example

Suppose an agent wants to buy a used 4-volume boxed set of *The Art of Computer Programming* by Don Knuth. Assume that the item arrives in any of the following conditions: *Very Good*, *Good* and *Acceptable*. Following are the two marketplaces available to the agent:

- *Market 1: Good* with probability 0.3, or *Very Good* with probability 0.7.
- *Market 2: Acceptable* with probability 0.3, or *Good* with probability 0.2, or *Very Good* with probability 0.5.

Let the utilities be $u_A = 100$, $u_G = 200$ and $u_{VG} = 300$. Then,

$$\begin{aligned} EU(\text{Market 1}) &= 0.3 \times 200 + 0.7 \times 300 = 270 \\ EU(\text{Market 2}) &= 0.3 \times 100 + 0.2 \times 200 + 0.5 \times 300 = 220 \end{aligned}$$

Prescription: Market 1 $\succ$ Market 2

Affine Transformation

Credit: Von Neumann and Morgenstern, 1947

Theorem 2

Let $E \in \mathcal{E}$ be any given subjective event. Suppose the agent's preferences satisfy Axioms 1-9, and let $u : \mathcal{X} \times \Omega \to [0,1]$ and $p : \mathcal{E} \to \Delta(\Omega)$ denote the state-independent utility function and conditional probability function as stated in Corollary 1. Let v be a state-independent utility function and q be a conditional probability function, which represent the preference ordering $\succsim_E$. Then, there exists numbers $a > 0$ and b such that

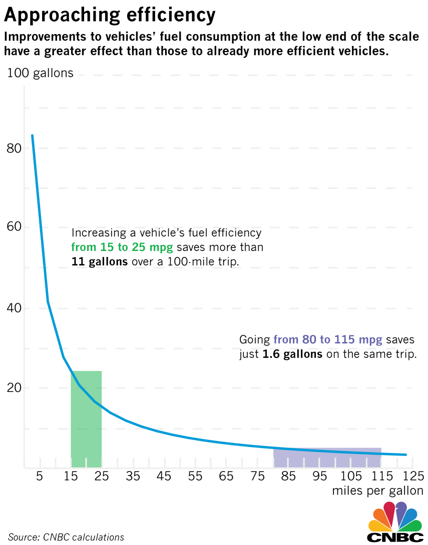
$$v(x) = au(x) + b, \text{ for all } x \in \mathcal{X}.$$

Types of Utility Functions

- **Risk Aversion:** A concave function of monetary value (wealth), i.e., $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y) \ \forall \ x, y \in \mathcal{X}$
- **Risk Seeking:** A convex function of monetary value (wealth), i.e., $u(\lambda x + (1-\lambda)y) < \lambda u(x) + (1-\lambda)u(y)$
- **Risk Neutral:** An affine function of monetary value (wealth).

Limitations of EUM: Preference Intensity

- $x_1 \succ x_2 \succ x_3 \succ x_4$ and $u(x_1) - u(x_2) > u(x_3) - u(x_4)$ $\not\Rightarrow$ change from x_2 to x_1 is more preferred than change from x_4 to x_3.
  - Ordinal utility does not measure both intensity and direction of preferences.
  - Need for *cardinal utility* to capture the strength of preferences.
  - Note: The notion of cardinal utility has a different meaning in measurement theory in psychology, which is irrelevant to our discussion.
  - Captures *information framing* effects



Limitations of EUM: Other Inconsistencies

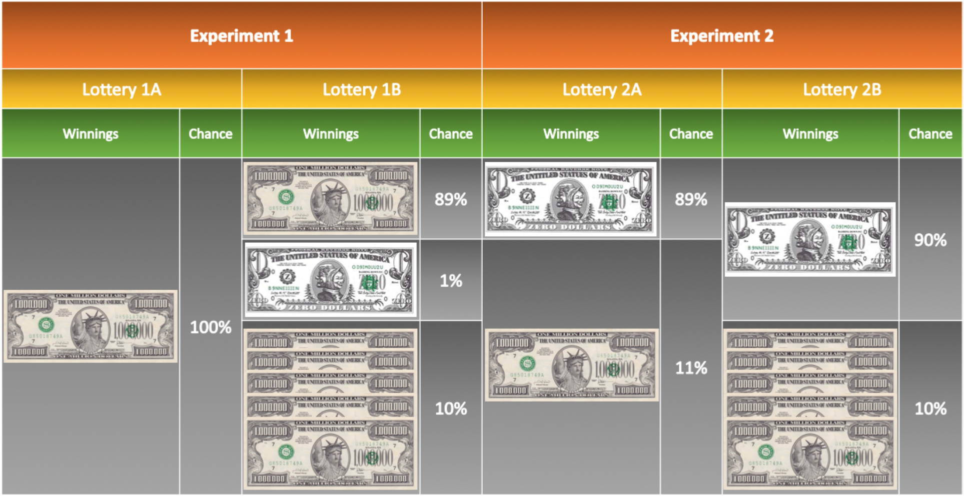
People's preferences does not necessarily satisfy Axioms 1-9.

- **Bounded Rationality:** Decisions under Limited Time/Memory/Attention $\Rightarrow$ *Satisficing*.
- **Behavioral Complexities:** Loss Aversion, Probability Weighting, Framing Effects and Preference Reversals, Anchoring Bias, Confirmation Bias, Polarization...
- **Prosociality:** Social Reputation/Pride

Allias Paradox

Credit: Maurice Allias, 1953

Pick one lottery from each experiment!



Allias Paradox (cont...)

Credit: Maurice Allias, 1953

Experiment 1				Experiment 2			
Lottery 1A		Lottery 1B		Lottery 2A		Lottery 2B	
Winnings	Chance	Winnings	Chance	Winnings	Chance	Winnings	Chance
	100%		89%		89%		90%
			1%		11%		10%
			10%				

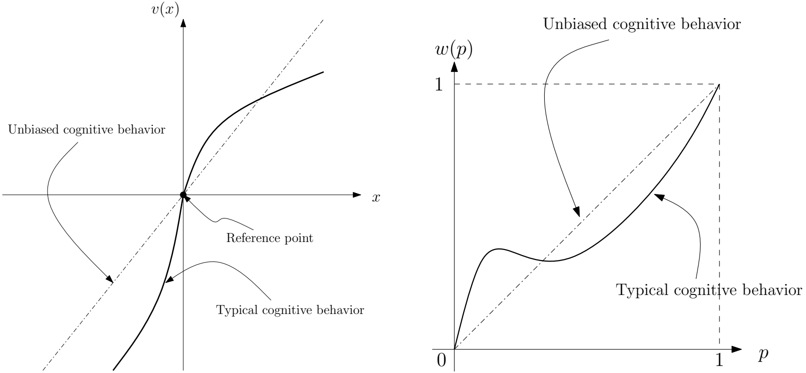
Usually $1A \succ 1B$ and $2B \succ 2A$ – *inconsistent with EUM!*

Prospect Theory – A Descriptive Model

Credit: Daniel Kahneman and Amos Tversky, 1979

Assuming that the choice experiment has only one state,

$$\text{maximize } V = \sum_{x \in \mathcal{X}} v(x) \cdot w(f(x))$$



Variants[2] of Expected Utility Theory

TABLE 1
NINE VARIANTS OF THE EXPECTED UTILITY MODEL

1.	$\Sigma\, p_i x_i$	Expected Monetary Value
2.	$\Sigma\, p_i v(x_i)$	Bernoullian Expected Utility (1738)
3.	$\Sigma\, p_i u(x_i)$	von Neumann-Morgenstern Expected Utility (1947)
4.	$\Sigma\, f(p_i) x_i$	Certainty Equivalence Theory (Schneeweiss, 1974; Handa, 1977; de Finetti, 1937)
5.	$\Sigma\, f(p_i) v(x_i)$	Subjective Expected Utility (Edwards, 1955)
6.	$\Sigma\, f(p_i) u(x_i)$	Subjective Expected Utility (Ramsey, 1931; Savage, 1954; Quiggin, 1980)
7.	$\Sigma\, w(p_i) x_i$	Weighted Monetary Value
8.	$\Sigma\, w(p_i) v(x_i)$	Prospect Theory (Kahneman and Tversky, 1979)
9.	$\Sigma\, w(p_i) u(x_i)$	Subjectively Weighted Utility (Uday Karmarkar, 1978)

Note: $v(x)$ denotes an interval scaled utility measure constructed under certainty; $u(x)$ denotes one constructed via lotteries.

Several other models have been proposed since 1982...

[2] Credit: P. J. H. Schoemaker, "The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations", J. Economic Literature, vol. 20, no. 2, pp. 529-563, June 1982.

Decisions Under Ignorance

Sometimes...

- Difficult to assess subjective probabilities
  - Unknown environments
  - Limited computational capabilities
  - Limited time/memory...
- Easy to eliminate some choices – the *dominated* ones!

How can we identify and eliminate the dominated choices?

Dominance

Let the agent have

- state-dependent utility function $u : \mathcal{X} \times \Omega \to \mathbb{R}$,
- a subjective probability $p(t)$ in a state $t \in \Omega$.

Let the agent does not make random decisions,
*i.e. lotteries are **deterministic**!*

Definition

The choice $y \in \mathcal{X}$ is dominating in a deterministic experiment only if

$$\sum_{t \in \Omega} p(t)u(y,t) \geq \sum_{t \in \Omega} p(t)u(x,t)$$

for all $x \in \mathcal{X}$.

Note: Dominance may hold true only in some $p(t) \in \Delta(\Omega)$.

Dominance (cont...)

For what distributions of p can a given choice $y \in \mathcal{X}$ be optimal for an agent with a state-dependent utility function u?

Theorem

Given $u : \mathcal{X} \times \Omega \to \mathbb{R}$ and given $y \in \mathcal{X}$, the set of all $p \in \Delta(\Omega)$ such that y is optimal is convex.

Example: Suppose $\mathcal{X} = \{\alpha, \beta, \gamma\}$, $\Omega = \{t_1, t_2\}$ and the corresponding utilities are as follows.

Decision	**State** t_1	**State** t_2
α	8	1
β	5	3
γ	4	7

Dominance (cont...)

Let $p(t_1) = p$. Then, we have $p(t_2) = 1 - p$.

- The decision α is optimal if and only if

$$8p + 1[1-p] \geq 5p + 3(1-p),$$
$$8p + 1[1-p] \geq 4p + 7(1-p).$$

In other words, $p \geq 0.6$.

- The decision β is optimal if and only if

$$5p + 3(1-p) \geq 8p + 1[1-p],$$
$$5p + 3(1-p) \geq 4p + 7(1-p).$$

But, this is an empty set!

So, β can never be optimal for any set of beliefs!

This is called a *strongly dominated* choice.

A Caveat...

*Just because α is the optimal choice in state t_1 and γ is the optimal choice is state t_2, we **cannot** claim that β (an intermediate choice) is dominated!*

Example: Consider the earlier example with the following utility table.

Decision	**State t_1**	**State t_2**
α	8	1
β	6	3
γ	4	7

Now, the decision β is optimal whenever $5/7 \leq p \leq 1/3$.

Dominance and Lotteries

Definition

A choice $y \in \mathcal{X}$ is strongly dominated by a lottery $f \in \Delta(\mathcal{X})$ if

$$\sum_{x \in \mathcal{X}} f(x|t) u(x,t) > u(y,t)$$

for all $t \in \Omega$.

Theorem

Given $u : \mathcal{X} \times \Omega \to \mathbb{R}$ and any choice $y \in \mathcal{X}$, there exists a lottery $f \in \Delta(\mathcal{X})$ such that y is strongly dominated by f, if and only if there does not exist any probability distribution $p \in \Delta(\Omega)$ such that y is optimal in a deterministic experiment.

Weak Dominance

Definition

A choice $y \in \mathcal{X}$ is weakly dominated by a lottery $f \in \Delta(\mathcal{X})$ if

$$\sum_{x \in \mathcal{X}} f(x|t)u(x,t) \geq u(y,t)$$

for all $t \in \Omega$, and there exists at least one state in Ω such that the above inequality is strict.

Example: Let $\mathcal{X} = \{\alpha, \beta\}$, $\Omega = \{t_1, t_2\}$ and utilities as given below.

Decision	**State** t_1	**State** t_2
α	5	3
β	5	1

Here, β is weakly dominated by α (due to the case where $p = 1.$).

Summary

- *St. Petersburg Paradox:* Why do we need utility functions?
- *Preference Axioms:* How does ideal agent's preferences look like?
- *Expected Utility Maximization:* Ideal agents maximize expected utilities.
- *Limitations:* People are not ideal agents.
- *Allais Paradox and Prospect Theory:* One example of a descriptive model.
- *Domination:* Decision making under ignorance