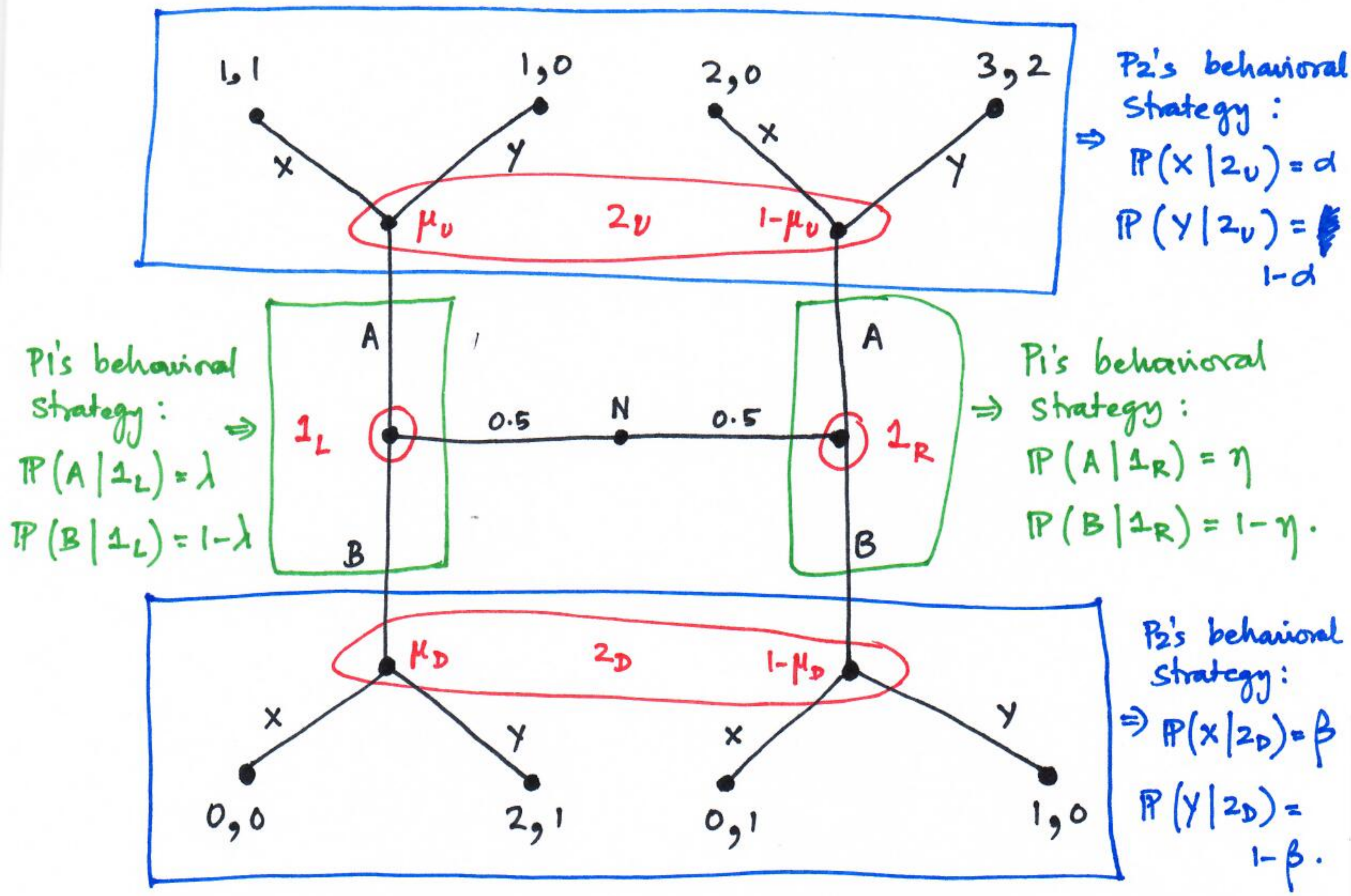
SIGNALING GAMES

#1

HANDOUT

Consider the following example
(also discussed in the class)



Let $\mu_U = \mathbb{P}(L \mid A)$ denote the belief at P2 (receiver) in information set 2_U.

Illy, $\mu_D = \mathbb{P}(L \mid B)$ denote the belief at P2 in information set 2_D.

Then, P2's conditional expected utilities are

$$u_2(x \mid 2_U) = 1 \cdot \mu_U + 0 \cdot (1-\mu_U) = \mu_U \quad \text{— (1a)}$$

$$u_2(y \mid 2_U) = 0 \cdot \mu_U + 2 \cdot (1-\mu_U) = 2 - 2\mu_U \quad \text{— (1b)}$$

$$u_2(x \mid 2_D) = 0 \cdot \mu_D + 1 \cdot (1-\mu_D) = 1 - \mu_D \quad \text{— (2a)}$$

$$u_2(y \mid 2_D) = 1 \cdot \mu_D + 0 \cdot (1-\mu_D) = \mu_D . \quad \text{— (2b)}$$

$\therefore$ P2's expected utilities (conditional) given their behavioral strategies are

$$\begin{aligned} u_2(2_U) &= \alpha \cdot u_2(x \mid 2_U) + (1-\alpha) \cdot u_2(y \mid 2_U) \\ &= \alpha \cdot \mu_U + (1-\alpha)(2-2\mu_U) \\ &= 3\alpha\mu_U - 2\alpha - 2\mu_U + 2 . \quad \text{— (3a)} \end{aligned}$$

$$\begin{aligned} u_2(2_D) &= \beta \cdot u_2(x \mid 2_D) + (1-\beta) \cdot u_2(y \mid 2_D) \\ &= \beta \cdot (1-\mu_D) + (1-\beta) \cdot \mu_D \\ &= \beta + \mu_D - 2\beta\mu_D . \quad \text{— (3b)} \end{aligned}$$

III^{ly}, P1's conditional expected utilities are (#3)

$$u_1(A \mid 1_L) = 1.\alpha + 1.(1-\alpha) = 1. \quad \text{— (4a)}$$

$$u_1(B \mid 1_L) = 0.\beta + 2.(1-\beta) = 2-2\beta. \quad \text{— (4b)}$$

$$u_1(A \mid 1_R) = 2\cdot\alpha + 3\cdot(1-\alpha) = 3-\alpha \quad \text{— (5a)}$$

$$u_1(B \mid 1_R) = 0.\beta + 1\cdot(1-\beta) = 1-\beta. \quad \text{— (5b)}$$

$\therefore$ P1's conditional expected utilities given their behavioral strategies are :

$$u_1(1_L) = \lambda.\, u_1(A \mid 1_L) + (1-\lambda).\, u_1(B \mid 1_L)$$

$$= \lambda.\, 1 + (1-\lambda)(2-2\beta)$$

$$= 2-2\beta-\lambda+2\lambda\beta. \quad \text{— (6a)}$$

$$u_1(1_R) = \eta.\, u_1(A \mid 1_R) + (1-\eta).\, u_1(B \mid 1_R)$$

$$= \eta\cdot(3-\alpha) + (1-\eta)(1-\beta)$$

$$= 1-\beta+2\eta-\alpha\eta+\eta\beta. \quad \text{— (6b)}$$

Consistency in P2's beliefs.

$$\mu_U = \frac{\mathbb{P}(L \mid 2_U)}{\mathbb{P}(L \mid 2_U) + \mathbb{P}(R \mid 2_U)}$$

$$= \frac{(0.5)\cdot\lambda}{(0.5)\cdot\lambda + (0.5)\cdot\eta} = \frac{\lambda}{\lambda+\eta} \quad \text{— } 7$$

lly, $$\mu_D = \frac{\mathbb{P}(L \mid 2_D)}{\mathbb{P}(L \mid 2_D) + \mathbb{P}(R / 2_D)}$$

$$= \frac{(0.5)(1-\lambda)}{(0.5)(1-\lambda) + (0.5)(1-\eta)} = \frac{1-\lambda}{2-\lambda-\eta}. \quad \text{— } 8$$

Substituting 7 in 3a, we obtain

$$u_2(2_U) = 2 - 2\alpha - 2.\left(\frac{\lambda}{\lambda+\eta}\right) + 3\alpha\left(\frac{\lambda}{\lambda+\eta}\right)$$

$$= \frac{2\eta - 2\alpha\eta + \alpha\lambda}{\lambda+\eta}. \quad \text{— } 9a$$

$|||^{ly}$,

$$u_2(2_D) = \beta + \frac{1-\lambda}{2-\lambda-\eta} - 2\beta\left(\frac{1-\lambda}{2-\lambda-\eta}\right)$$

$$= \frac{1-\lambda+\beta\lambda-\beta\eta}{2-\lambda-\eta} \quad \text{— (9b)}$$

P1's Sequential Rationality

(6a) $\Rightarrow u_1(1_L) = 2(1-\beta) + \lambda(2\beta-1)$

(a linear function of λ)

(6b) $\Rightarrow u_1(1_R) = (1-\beta) + \eta(2-\alpha+\beta)$

(a linear function of η)

Note that the coeff. of η in $u_1(1_R)$ is always +ve.

$\Rightarrow u_1(1_R)$ is max. when $\boxed{\eta = 1.}$ — (10)

$$\Rightarrow u_1(1_R)\Big|_{\eta=1} = 3-\alpha.$$

However,

$u_1(1_L)$ is maximized by choosing

$$\lambda = \begin{cases} 1 & \text{if } 2\beta-1 > 0 \\ [0,1] & \text{if } 2\beta-1 = 0 \\ 0 & \text{if } 2\beta-1 < 0. \end{cases}$$

P.T.O.

In other words,

$$\boxed{\lambda = \begin{cases} 1 \ ; \text{ if } \beta > \frac{1}{2} \\ [0,1] \ ; \text{ if } \beta = 1/2 \\ 0 \ ; \text{ if } \beta < \frac{1}{2} . \end{cases}} \quad \text{— (11)}$$

<u>P2's Sequential Rationality</u>

(9a) $\Rightarrow \quad u_2(2_U) = \dfrac{2\eta}{\lambda+\eta} + \alpha\left[\dfrac{\lambda - 2\eta}{\lambda+\eta}\right]$

(a linear fn. of α)

(9b) $\Rightarrow \quad u_2(2_D) = \dfrac{1-\lambda}{2-\lambda-\eta} + \beta\left[\dfrac{\lambda-\eta}{2-\lambda-\eta}\right]$

(a linear fn. of β).

Since we know $\eta = 1$,

the coeff. of α in $u_2(2_U)$ is -ve.

$\Rightarrow \quad u_2(2_U)$ is maximized when $\boxed{\alpha = 0}$.

However, the coeff. of β in $u_2(2_D)$ is either -ve (if $\lambda < 1$)
or equal to zero (if $\lambda = 1$).

$$\Rightarrow \quad \boxed{\beta = \begin{cases} 0 & \text{if } \lambda < 1 \\ [0,1] & \text{if } \lambda = 1. \end{cases}} \quad \text{— (12)} .$$

In Summary, we have

$$\alpha = 0 \qquad\qquad \eta = 1$$

$$\beta = \begin{cases} 0 & \text{if } \lambda < 1 \\ [0,1] & \text{if } \lambda = 1 \end{cases} \qquad \lambda = \begin{cases} 1 & \text{if } \beta > \frac{1}{2} \\ [0,1] & \text{if } \beta = \frac{1}{2} \\ 0 & \text{if } \beta < \frac{1}{2} \end{cases}.$$

This results in two equilibria :

Equilibrium 1 : $\alpha = 0,\ \beta = 0,\ \eta = 1,\ \lambda = 0$

i.e. $\boxed{(BA, YY)}$

with $\boxed{\mu_U = \frac{0}{0+1} = 0}$ and $\boxed{\mu_D = \frac{1}{2-1} = 1}$.

This is a <u>Separating equilibrium</u>.

Equilibrium 2 : $\alpha = 0,\ \beta > \frac{1}{2},\ \eta = 1,\ \lambda = 1.$

i.e. $\left(AA, \left(Y, \mathbb{P}(X|2_D) > \frac{1}{2} \right) \right)$

This is a <u>pooling equilibrium</u>. with $\boxed{\mu_U = \frac{1}{1+1} = \frac{1}{2}}$ and $\mu_D = \frac{1-0}{2-1-1} \rightarrow$ <u>indeterm.</u>

This can also cause issues in 9b as well.

So, going back to 3b,

max. $u_2(2_D) = \mu_D + \beta(1 - 2\mu_D)$ for some $\beta > \frac{1}{2}$ makes sense only when $1 - 2\mu_D \geq 0$

$\Rightarrow \beta = 1$ and $\boxed{\mu_D \leq \frac{1}{2}} \Rightarrow \boxed{(AA, YX)}$