Missouri University of Science & Technology — Department of Computer Science

Fall 2022 — **CS 5408: Game Theory for Computing**

Solutions to Homework 4a: Dynamic Games

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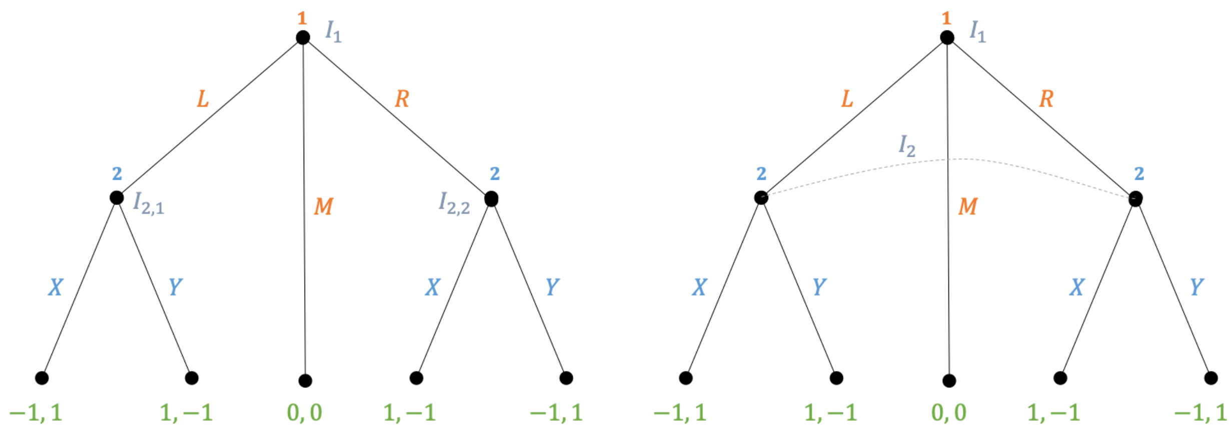
Problem 1 Complete Extensive Games *4 pts.*

Consider the following modified matching pennies game, played in extensive form, where Prisoner 1 plays first, followed by Prisoner 2. The main difference from the traditional mathcing pennies is that Player 1 can decide whether to play this game, or not. If he decides not to play, both players get nothing.



(a) Find the subgame perfect equilibrium for this game, when Player 2 can perfectly observe Player 1's choices as in the left figure.

(b) Find behavioral equilibria for this game, when Player 2 cannot observe Player 1's choices as in the right figure.

Solution:

(a) Backward induction is used to compute subgame perfect Nash equilibrium, as shown in Figure 1. In the first stage, Nash equilibrium for subgames rooted at nodes 2 and 3 are first computed. At the end of the first stage, the values at nodes 2 and 3 are both updated to $(-1, 1)$. In the second stage, the Nash equilibrium for the subgame rooted at node 1 is evaluated and the value of node 1 is updated as $(0, 0)$.

Therefore, SPNE is $(P_1 : M, P_2 : X/Y)$. □

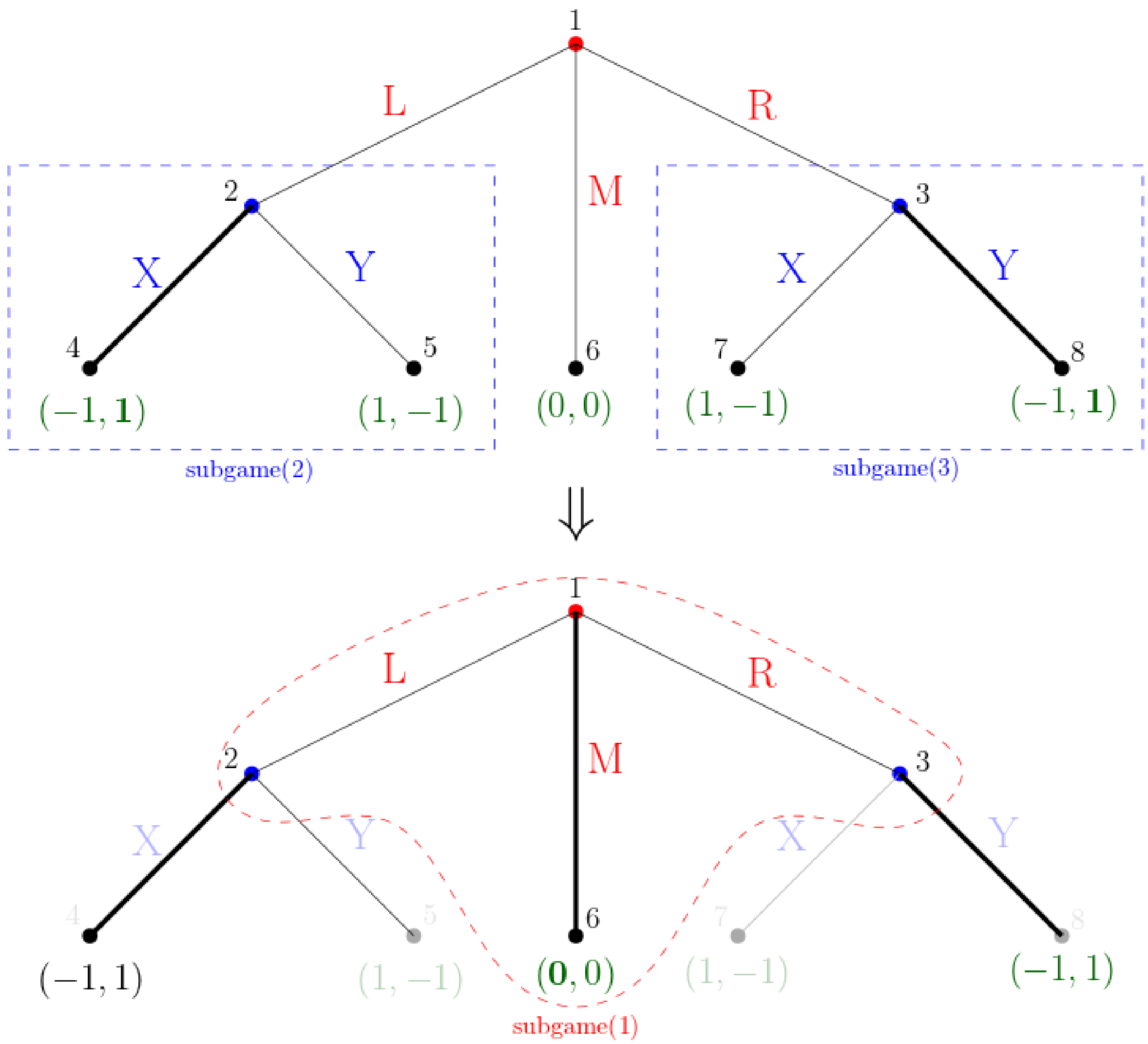


Figure 1: Stages of Backward Induction to compute SPNE

(b) Let P_2's behavioral strategy in I_2 be $\{X : \alpha, Y : 1-\alpha\}$. Also, assume that P_2 constructs a belief $\mu = \mathbb{P}(L|I_2)$ regarding being in the left node in I_2. Then, P_2's conditional expected utilities are given by

$$\begin{aligned} u_2(X|I_2) &= \mu \cdot 1 + (1-\mu) \cdot (-1) &= 2\mu - 1 \\ u_2(Y|I_2) &= \mu \cdot (-1) + (1-\mu) \cdot 1 &= 1 - 2\mu \end{aligned} \tag{1}$$

Therefore, the expected utility at P_2 due to the behavioral strategy $\{X : \alpha, Y : 1-\alpha\}$ is given by

$$u_2(I_2) = \alpha \cdot u_2(X|I_2) + (1-\alpha) \cdot u_2(Y|I_2) = (1-2\alpha)(1-2\mu). \tag{2}$$

Similarly, P_1's expected utilities are given by

$$\begin{aligned}
u_1(L) &= \alpha \cdot (-1) + (1-\alpha) \cdot 1 &= 1 - 2\alpha, \\
u_1(R) &= \alpha \cdot 1 + (1-\alpha) \cdot (-1) &= 2\alpha - 1, \\
u_1(M) &= 0.
\end{aligned} \tag{3}$$

Note that P_1's sequential rationality is satisfied by the following best-response strategy:

- If $\alpha > \dfrac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses R.
- If $\alpha < \dfrac{1}{2}$, then $u_1(L) < u_1(M) < u_1(R) \Rightarrow P_1$ chooses L.
- If $\alpha = \dfrac{1}{2}$, then $u_1(L) = u_1(M) = u_1(R) \Rightarrow P_1$'s preference order is $L \sim M \sim R$.

Similarly, P_1's sequential rationality is satisfied by the following best-response strategy:

- If $\mu > \dfrac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 1$.
- If $\mu < \dfrac{1}{2}$, then $u_2(I_2)$ is maximized when $\alpha = 0$.
- If $\mu = \dfrac{1}{2}$, then $u_2(I_2) = u_2(M) = 0 \Rightarrow P_2$'s preference order is $X \sim Y$.

Now, P_2's consistency is guaranteed if

- If $\alpha < \dfrac{1}{2}$, then P_1 chooses $L \Rightarrow \mu = 1$.

  But, this is a <u>violation</u> to P_2's sequential rationality since P_2 chooses $\alpha = 1$ if $\mu > \frac{1}{2}$.
- If $\alpha > \dfrac{1}{2}$, then P_1 chooses $R \Rightarrow \mu = 0$.

  But, this is a <u>violation</u> to P_2's sequential rationality since P_2 chooses $\alpha = 0$ if $\mu < \frac{1}{2}$.

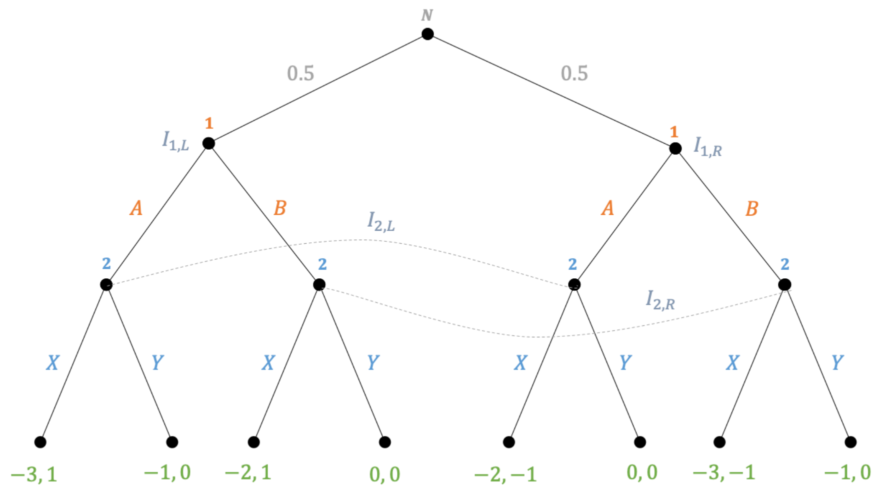
This leads us to the behavioral equilibrium, which is

- P_1 chooses M,
- P_2 chooses $\{X : \frac{1}{2}, Y : \frac{1}{2}\}$, with $\mu = \frac{1}{2}$.

$\square$

Problem 2 Perfect Bayesian Equilibrium *3 pts.*

Prove that there is no separating equilibrium in the following two-player signaling game (as depicted in the figure below), where the player set is $\mathcal{N} = \{1, 2\}$, the choice sets at the corresponding players are $\mathcal{C}_1 = \{A, B\}$ and $\mathcal{C}_2 = \{X, Y\}$ respectively. Assume that Player 1 can take two types $\{L, R\}$, and Player 2's belief about Player 1's type is uniformly distributed across types.



Solution: Let the pure strategy at Player 1 be denoted by two letters, where the first letter corresponds to the strategy chosen in the information set $I_{1,L}$ and the second letter represents the strategy chosen in the information set $I_{1,R}$. For example, a pure strategy AB means that the sender chooses A in $I_{1,L}$ and B in $I_{1,R}$.

Note that Player 1 only has two separating strategies: AB and BA. Let us consider each of these strategies on a case-by-case basis:

Case 1 (AB): Since this is a separating strategy, the receiver clearly knows the information set he/she is in. For example, if the receiver observes a signal A, then he/she is on the left node of the information set $I_{2,L}$. In such a case, the receiver will choose X since $u_2(X|AB, I_{2,L}) = 1 > 0 = u_2(Y|AB, I_{2,L})$. Similarly, in $I_{1,R}$, if the sender chooses B, the receiver will always choose Y since $u_2(Y|AB, I_{2,R}) = 0 > -1 = u_2(X|AB, I_{2,R})$. In other words, the receiver's best response to AB is XY. However, sequential rationality is satisfied if the sender's best response to XY is also AB. However, if receiver always chooses X in $I_{2,L}$ and Y in $I_{2,R}$, then sender will always choose B at $I_{1,L}$ since $u_1(B|XY, I_{1,L}) = 0 > -3 = u_1(A|XY, I_{1,L})$. In other words, sequential rationality is violated for the separating strategy AB.

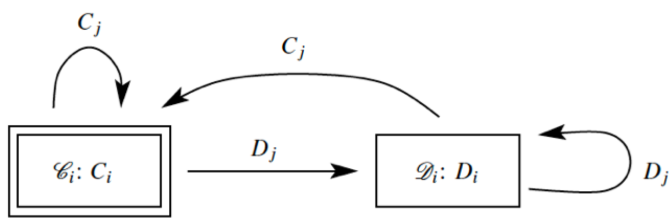
Case 1 (BA): Since $u_2(X|BA, I_{1,L}) = 1 > 0 = u_2(Y|BA, I_{1,L})$ and $u_2(Y|BA, I_{1,R}) = 0 > -1 = u_2(X|BA, I_{1,R})$, the receiver's best response to BA is XY. However, sequential rationality is satisfied if the sender's best response to XY is also BA. However, in $I_{1,R}$, the sender always chooses B since $u_1(B|XY, I_{1,R}) = -1 > -2 = u_1(A|XY, I_{1,R})$. This is a violation of sequential rationality condition too.

In other words, since separating strategies violate sequential rationality condition, this game does not have a separating equilibrium. $\square$

Problem 3 Repeated Games ***3 pts.***

Consider the following repeated prisoner's dilemma game, where players play the game over an infinite time horizon. Prove that Tit-for-Tat strategy (given below) is a Nash equilibrium to this game, only when the discounting factor $\beta \geq \frac{1}{2}$.

		Prisoner 2	
		C_2	D_2
Prisoner 1	C_1	2, 2	0, 3
	D_1	3, 0	1, 1



Solution:

Assuming that Player $-i$ follows Tit-for-Tat, Player i's responses can be summarized by the following four classes of strategy profile sequences of length T:

- **CASE (a):**

$$\begin{array}{lrcccccc} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T)} & \cdots \end{array}$$

- **CASE (b):**

$$\begin{array}{lrcccccccc} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & D_i^{(T+1)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & \cdots \end{array}$$

- **CASE (c):**

$$\begin{array}{lrccccccccc} \text{Player} & i: & C_i^{(1)} & C_i^{(2)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & C_i^{(T+1)} & C_i^{(T+2)} & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & C_{-i}^{(2)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & C_{-i}^{(T+2)} & \cdots \end{array}$$

- **CASE (d):**

$$\begin{array}{lrcccccccccc} \text{Player} & i: & C_i^{(1)} & \cdots & C_i^{(T-1)} & D_i^{(T)} & D_i^{(T+1)} & \cdots & D_i^{(T+k-1)} & C_i^{(T+k)} & & \cdots \\ \text{Player} & -i: & C_{-i}^{(1)} & \cdots & C_{-i}^{(T-1)} & C_{-i}^{(T)} & D_{-i}^{(T+1)} & \cdots & D_{-i}^{(T+k-1)} & D_{-i}^{(T+k)} & C_{-i}^{(T+k+1)} & \cdots \end{array}$$

Each of these classes of strategy profile sequences generates the following discounted utilities at Player i:

$$\begin{aligned}
u_i^{(a)} &= \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(b)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \;+\; \beta^{T-1} \cdot 3 \;+\; \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
u_i^{(c)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \;+\; \beta^{T-1} \cdot 3 \;+\; \beta^{T} \cdot 0 \;+\; \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
u_i^{(d)} &= \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \;+\; \beta^{T-1} \cdot 3 \;+\; \sum_{t=T+1}^{T+k-1} \beta^{t-1} \cdot 1 \;+\; \beta^{T+k-1} \cdot 0 \;+\; \sum_{t=T+k+1}^{\infty} \beta^{t-1} \cdot 2
\end{aligned} \tag{4}$$

Note that Case (a) corresponds to Tit-for-Tat strategy at Player i. In other words, Tit-for Tat at Player i is the best response to Tit-for Tat at Player $-i$ if

$$u_i^{(a)} \geq u_i^{(b)}, \tag{5}$$

$$u_i^{(a)} \geq u_i^{(c)}, \tag{6}$$

$$u_i^{(a)} \geq u_i^{(d)}. \tag{7}$$

Substituting Equation (1) in inequalities (2)-(4), we obtain

$$\begin{aligned}
& \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \;\geq\; \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \;+\; \beta^{T-1} \cdot 3 \;+\; \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow \quad & \sum_{t=T}^{\infty} \beta^{t-1} \cdot 2 \;\geq\; \beta^{T-1} \cdot 3 \;+\; \sum_{t=T+1}^{\infty} \beta^{t-1} \cdot 1 \\
\Rightarrow \quad & \sum_{t=T+1}^{\infty} \beta^{t-1} \;\geq\; \beta^{T-1} \\
\Rightarrow \quad & \beta^{T} \cdot \frac{1}{1-\beta} \;\geq\; \beta^{T-1} \\
\Rightarrow \quad & \beta^{T-1}(2\beta - 1) \;\geq\; 0, \quad \text{or} \quad \beta \geq \frac{1}{2},
\end{aligned} \tag{2a}$$

$$\begin{aligned}
& \sum_{t=1}^{\infty} \beta^{t-1} \cdot 2 \;\geq\; \sum_{t=1}^{T-1} \beta^{t-1} \cdot 2 \;+\; \beta^{T-1} \cdot 3 \;+\; \sum_{t=T+2}^{\infty} \beta^{t-1} \cdot 2 \\
\Rightarrow \quad & \beta^{T-1} \cdot (2\beta - 1) \;\geq\; 0, \quad \text{or} \quad \beta \geq \frac{1}{2},
\end{aligned} \tag{3a}$$

$$
\begin{aligned}
\sum_{t=1}^{\infty}\beta^{t-1}\cdot 2 \;&\geq\; \sum_{t=1}^{T-1}\beta^{t-1}\cdot 2 \;+\; \beta^{T-1}\cdot 3 \;+\; \sum_{t=T+1}^{T+k-1}\beta^{t-1}\cdot 1 \;+\; \sum_{t=T+k+1}^{\infty}\beta^{t-1}\cdot 2 \\
\Rightarrow \qquad 2\beta^{T-1}\cdot\frac{1-\beta^{k+1}}{1-\beta} \;&\geq\; 3\beta^{T-1}+\beta^{T}\cdot\frac{1-\beta^{k-1}}{1-\beta} \\
\Rightarrow \quad 1-2\beta-\beta^{k}+2\beta^{k+1} \;&\leq\; 0 \\
\Rightarrow \qquad (1-2\beta)(1-\beta^{k}) \;&\leq\; 0, \quad \text{or} \quad \beta\geq\frac{1}{2}.
\end{aligned} \tag{4a}
$$

Since Inequalities (2a)-(4a) all hold true when $\beta \geq \frac{1}{2}$, Tit-for-Tat is a best response strategy for Player i against a Tit-for-Tat strategy adopted by Player $-i$. Since the analysis holds true for both $i = 1, 2$, Tit-for-Tat is a Nash equilibrium if $\beta \geq \frac{1}{2}$.

□