Topic 5: Coalitional Games

Israeli election 2019: Preliminary results

- Blue and White (35 seats)
- Labour (6), Meretz (4)
- Hadash Taal (6), Balad UAL (4)
- Likud (35), Yisrael Beiteinu (5), Rightist Union (5)
- Kulanu (4)
- Shas (8), United Torah Judaism (8)

Netanyahu’s potential coalition partners won 65 of 120 seats

Source: Times of Israel, Jerusalem Post and Haaretz (95% of votes counted)
Outcomes & Objectives

▶ Be proficient in solving coalitional games
  ▶ Model player’s rationality in forming coalitions via defining a value of a given coalition.
  ▶ Identify some useful subclasses of games which produces some special coalitions.
  ▶ Develop a solution concept called *Shapley value* to distribute a coalition’s value in a fair manner.
  ▶ Develop a solution concept called *core* that identifies a stable coalition structure in the game.
Lloyd Shapley

Shapley was the greatest game theorist of all time.

– Robert Aumann
Applications of Coalitional Games

▶ **Political Coalitions:** Parties form coalitions if the elections did not result in one party with a majority votes. Coalitional governments resolve such concerns. However, the question is which coalitions form stable governments.

▶ **Cost Sharing for Network Design:** Users benefit from being connected to a server. So they have to build up a broadcast tree. However, it costs to maintain the server/network and the question is how to share the costs.

▶ **Queue Management:** Multiple users want to route traffic through a switch, which has a flow dependent delay (cost). The queueing delay cost has to be shared among the users.
Coalitional Game: An Overview

- How to form stable teams?
- Coalition Value as Outcomes of the Game
- How to distribute coalition value in a fair manner?

Players → Coalitions → Noncooperative Game on Coalition Structure → Distribution
Coalitions and Transferable Utilities

Definition

Given a set of players \( \mathcal{N} = \{1, \cdots, N\} \), a **coalition** is a subset of \( \mathcal{N} \). Furthermore, a **grand coalition** is the set of all players \( \mathcal{N} \).

Definition

A **characteristic function game** \( \Gamma \) is a pair \( (\mathcal{N}, v) \), where \( \mathcal{N} \) is the set of players, and \( v : 2^\mathcal{N} \to \mathbb{R} \) is a **characteristic function**, which assigns each coalition \( C \subseteq \mathcal{N} \), some real value \( v(C) \).

Definition

A characteristic function game \( \Gamma = (\mathcal{N}, v) \) is a **transferable utility game**, if the value of any coalition \( v(C) \) can be distributed amongst the members in \( C \) in any way that the members of \( C \) choose.

**Standard Assumptions:**

- The value of a empty coalition is 0.
- \( v(C) \geq 0 \), for any \( C \subseteq \mathcal{N} \).
Example

A fictional country $X$ has a 101-member parliament, where each representative belongs to one of the three parties:

- Liberal ($L$): 40 representatives
- Moderate ($M$): 31 representatives
- Conservative ($C$): 30 representatives

The parliament needs to decide how to allocate $1bn of discretionary spending, and each party has its own preferred way of using this money. The decision is made by a simple majority vote, and we assume that all representatives vote along the party lines.

Parties can form coalitions; a coalition has value $1bn if it can win the budget vote no matter what the other parties do, and value 0 otherwise.

This situation can be modeled as a three-player characteristic function game, where the set of players is $\mathcal{N} = L, M, C$ and the characteristic function is given by

$$v(C) = \begin{cases} 
0, & \text{if } |C| \leq 1, \\
10^9, & \text{otherwise}.
\end{cases}$$
Coalition Structure

Definition

Given a characteristic function game $\Gamma = (\mathcal{N}, v)$, a coalition structure $\mathcal{C}$ is a partition of $\mathcal{N}$. In other words, $\mathcal{C}$ is a collection of non-empty subsets $\{C_1, ..., C_K\}$ such that

$\bigcup_{k \in \{1, ..., K\}} C_k = \mathcal{N}$, and

$C_i \cap C_j = \emptyset$, for any $i, j \in \{1, ..., K\}$ such that $i \neq j$.

Definition

A vector $u = \{u_1, \cdots, u_N\} \in \mathbb{R}^N$ is the utility profile for a coalition structure $\mathcal{C} = \{C_1, \cdots, C_K\}$ over $\mathcal{N}$ if

- **Non-Negativity:** $u_i \geq 0$ for all $i \in \mathcal{N}$, and

- **Feasibility:** $\sum_{i \in C_k} u_i \leq v(C_k)$ for any $k \in \{1, \cdots, K\}$.
Definition

The outcome of a game $\Gamma$ is a pair $(C, u)$.

Definition

An outcome $(C, u)$ is efficient, if all the utilities are distributed amongst the coalition members, i.e.

$$\sum_{i \in C_k} u_i = v(C_k), \text{ for all } k = 1, \cdots, K.$$ 

Definition

The social welfare of a coalition structure $C$ is

$$v(C) = \sum_{k=1}^{K} v(C_k)$$
Individual Rationality and Imputation

**Definition**

A player $i$ is said to be *individually rational* in an outcome $(C, u)$, if

$$u_i \geq v(\{i\}),$$

where $v(\{i\})$ is the value of the coalition $\{i\}$, which only contains the $i^{th}$ player.

**Definition**

A outcome $(C, u)$ is said to be an *imputation*, if it is efficient, and if every player is individually rational within itself.

- Each player weakly prefers being in the coalition structure, than being on his/her own.
- Group deviations $\Rightarrow$ Stability of Coalitions (covered later)
Monotone Games

**Definition**

A characteristic function game \( \Gamma = \{N, v\} \) is said to be **monotone** if it satisfies \( v(C) \leq v(D) \), for every pair of coalitions \( C, D \subseteq N \), such that \( C \subseteq D \).

- Most games are monotone!
- However, non-monotonicity may arise because
  - some players intensely dislike each other, or
  - communication costs increase nonlinearly with coalition size.

**Example:** Three commuters can share a taxi. Individual journey costs: \( P_1 : 6 \), \( P_2 : 12 \), \( P_3 : 42 \). Then, the following characteristic function results in a monotone game:

\[
v_1(C) = \begin{cases} 
6 & \text{if } C = \{1\} \\
12 & \text{if } C = \{2\} \\
42 & \text{if } C = \{3\} \\
12 & \text{if } C = \{1, 2\} \\
42 & \text{if } C = \{1, 3\} \\
42 & \text{if } C = \{2, 3\} \\
42 & \text{if } C = \{1, 2, 3\}.
\end{cases}
\]
Superadditive Games

**Definition**

A characteristic function game $\Gamma = \{N, v\}$ is said to be **superadditive** if it satisfies $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.

**Proposition**

If a superadditive game $\Gamma = \{N, v\}$ has a non-negative characteristic function $v$, then $\Gamma$ is monotone.

*Proof:* For any pair of coalitions $C \subseteq D$, we have

$$v(C) \leq v(D) - v(D - C) \leq v(D).$$

$\square$

- Monotonicity $\not\implies$ superadditivity. (Example: $v(C) = \log |C|$.)
- Always profitable for two groups to join forces $\Rightarrow$ Grand Coalition.
- *Anti-trust* or *anti-monopoly* laws $\Rightarrow$ Non-superadditive games.
Superadditive Games: Example

Consider the same taxi example:

- Three commuters can share a taxi. Individual journey costs: $P_1 : 6$, $P_2 : 12$, $P_3 : 42$.

- Then, $v_1(C)$ is not superadditive.

- However, the following characteristic function results in a superadditive game:

$$v_2(C) = \begin{cases} 
6 & \text{if } C = \{1\} \\
12 & \text{if } C = \{2\} \\
42 & \text{if } C = \{3\} \\
18 & \text{if } C = \{1, 2\} \\
48 & \text{if } C = \{1, 3\} \\
55 & \text{if } C = \{2, 3\} \\
80 & \text{if } C = \{1, 2, 3\}.
\end{cases}$$
Convex Games

Definition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is said to be \textit{convex} if the characteristic function $v$ is supermodular, i.e., it satisfies $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$ for every pair of coalitions $C, D \subseteq \mathcal{N}$.

Proposition

A characteristic function game $\Gamma = \{\mathcal{N}, v\}$ is convex, if and only if, for every pair of coalitions $C, D$ such that $C \subset D$, and for every player $i \in \mathcal{N} - D$, we have

$$v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D)$$

- Players become more useful if they join bigger coalitions.
- Convexity $\Rightarrow$ Superadditivity.
- However, the converse may not be true!

\textit{3-player majority game:} Consider a game $\Gamma = (\mathcal{N}, v)$, where $\mathcal{N} = \{1, 2, 3\}$, and $v(C) = 1$ if $|C| \geq 2$, and $v(C) = 0$ otherwise. This game is superadditive. On the other hand, for $C = \{1, 2\}$ and $D = \{2, 3\}$, we have $v(C) = v(D) = 1$, $v(C \cup D) = 1$, $v(C \cap D) = 0$. 

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## Simple Games

### Definition

A characteristic function game \( \Gamma = \{ \mathcal{N}, v \} \) is said to be **simple** if it is monotone and its characteristic function only takes values 0 and 1, i.e. \( v(C) \in \{0, 1\} \), for any \( C \subseteq \mathcal{N} \).

- \( v(C) = 1 \) ⇒ Winning Coalition.
- \( v(C) = 0 \) ⇒ Loosing Coalition.

### Claim

A simple game \( \Gamma = \{ \mathcal{N}, v \} \) is superadditive, only if the complement of every winning coalition looses.
Solution Concepts

Outcomes can be evaluated based on two sets of criteria:

- **Fair Distribution:** How well each agent’s payoff reflects his/her contribution?
  - Shapley Value
  - Banzhaf Index

- **Coalition Stability:** What are the incentives for the agents to stay in the coalition structure?
  - Stable Set
  - Core
  - Nucleolus
  - Bargaining Set
Fair Distribution: Shapley’s Axioms

Let $u_i^\Gamma$ denote the allocation (utility) to the $i^{th}$ player in a game $\Gamma = \{N, v\}$. Then, we desire the following four properties:

- **Efficiency**: Distribute the value of grand coalition to all agents, i.e.
  \[
  \sum_{i \in N} u_i^\Gamma = v(N).
  \]

- **Dummy Player**: If a player $i$ does not contribute to any coalition in $\Gamma$, then
  \[
  u_i^\Gamma = 0.
  \]

- **Symmetry**: If two players $i$ and $j$ contribute equally to each coalition in $\Gamma$, then
  \[
  u_i^\Gamma = u_j^\Gamma.
  \]

- **Additivity**: If the same set of players are involved in two coalitional games $\Gamma_1 = (N, v_1)$ and $\Gamma_2 = (N, v_2)$, if we define $\Gamma = \Gamma_1 + \Gamma_1 = (N, v_1 + v_2)$, then for every player $i$, we have
  \[
  u_i^\Gamma = u_i^{\Gamma_1} + u_i^{\Gamma_2}.
  \]
Finding a Fair Distribution...

Assume we have a superadditive game, which results in a grand coalition!

- Agent’s allocation is proportional to his/her contribution in $v(N)$.
- Idea: As each agent joins to form the grand coalition, compute how much the value of the coalition increases, i.e., allocate $u_i = v(N) - v(N - \{i\})$ to player $i$.

This contribution is evaluated when the player is the last inclusion in $N$.

But, what about players who joined the coalition before the last player?

Let $\Pi_N$ denote the set of all permutations of $N$, i.e., one-to-one mappings from $N$ to itself. Given a permutation $\pi \in \Pi_N$, we denote by $S_\pi(i)$ the set of all predecessors of $i$ in $\pi$, i.e., we set

$$S_\pi(i) = \{ j \in N \mid \pi(j) < \pi(i) \}.$$

Example: If $N = \{1, 2, 3\}$, we have

$$\Pi_N = \{\{1, 2, 3\}, \{1, 3, 2\}, \{2, 1, 3\}, \{2, 3, 1\}, \{3, 1, 2\}, \{3, 2, 1\}\}.$$

Then, if $\pi = \{2, 1, 3\}$, we have

$$S_\pi(2) = \emptyset \quad S_\pi(1) = \{2\} \quad S_\pi(3) = \{1, 2\}.$$
Shapley Value

**Definition**

The *marginal contribution* of an agent $i$ with respect to a permutation $\pi$ in a game $\Gamma = (\mathcal{N}, v)$ is given by

$$\Delta^\Gamma_\pi(i) = v[S_\pi(i) \cup \{i\}] - v[S_\pi(i)].$$

**Definition**

Given a characteristic function game $\Gamma = (\mathcal{N}, v)$ with $|\mathcal{N}| = N$, the *Shapley value* of an agent $i \in \mathcal{N}$ is given by

$$u_i(\Gamma) = \frac{1}{N!} \sum_{\pi \in \Pi_N} \Delta^\Gamma_\pi(i).$$

**Theorem**

Shapley’s axioms *uniquely* characterize Shapley value. In other words, Shapley value is the only fair distribution scheme that satisfies all the Shapley’s axioms.
Shapley Value: Example

Consider the same ridesharing example, as stated earlier.

- Three commuters can share a taxi.
- Individual journey costs: $P_1 : 6$, $P_2 : 12$, $P_3 : 42$.
- The characteristic function is

$$v_1(C) = \begin{cases} 
6 & \text{if } C = \{1\} \\
12 & \text{if } C = \{2\} \\
42 & \text{if } C = \{3\} \\
12 & \text{if } C = \{1, 2\} \\
42 & \text{if } C = \{1, 3\} \\
42 & \text{if } C = \{2, 3\} \\
42 & \text{if } C = \{1, 2, 3\}.
\end{cases}$$

Permutation set $\Pi_N = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$, where

$$\pi_1 = \{1, 2, 3\}, \quad \pi_2 = \{1, 3, 2\}, \quad \pi_3 = \{2, 1, 3\},$$
$$\pi_4 = \{2, 3, 1\}, \quad \pi_5 = \{3, 1, 2\}, \quad \pi_6 = \{3, 2, 1\}.$$
Shapley Value: Example (cont...)

Given

\[ \pi_1 = \{1, 2, 3\}, \quad \pi_2 = \{1, 3, 2\}, \quad \pi_3 = \{2, 1, 3\}, \quad \pi_4 = \{2, 3, 1\}, \quad \pi_5 = \{3, 1, 2\}, \quad \pi_6 = \{3, 2, 1\}, \]

and

\[ v_1(C) = \begin{cases} 
6 & \text{if } C = \{1\} \\
12 & \text{if } C = \{2\} \\
42 & \text{if } C = \{3\} \\
12 & \text{if } C = \{1, 2\} \\
42 & \text{if } C = \{1, 3\} \\
42 & \text{if } C = \{2, 3\} \\
42 & \text{if } C = \{1, 2, 3\}. 
\end{cases} \]

Marginal contributions:

- **\( \pi_1 \):** \( \Delta_{\Gamma}^1(1) = 6, \Delta_{\Gamma}^1(2) = 6, \Delta_{\Gamma}^1(3) = 30 \)
- **\( \pi_2 \):** \( \Delta_{\Gamma}^2(1) = 6, \Delta_{\Gamma}^2(2) = 0, \Delta_{\Gamma}^2(3) = 36 \)
- **\( \pi_3 \):** \( \Delta_{\Gamma}^3(1) = 0, \Delta_{\Gamma}^3(2) = 12, \Delta_{\Gamma}^3(3) = 30 \)
- **\( \pi_4 \):** \( \Delta_{\Gamma}^4(1) = 0, \Delta_{\Gamma}^4(2) = 12, \Delta_{\Gamma}^4(3) = 30 \)
- **\( \pi_5 \):** \( \Delta_{\Gamma}^5(1) = 0, \Delta_{\Gamma}^5(2) = 0, \Delta_{\Gamma}^5(3) = 42 \)
- **\( \pi_6 \):** \( \Delta_{\Gamma}^6(1) = 0, \Delta_{\Gamma}^6(2) = 0, \Delta_{\Gamma}^6(3) = 42 \)

Shapley value:

- **\( u_1(\Gamma) \):** 
  \[ u_1(\Gamma) = \frac{1}{6} \sum_{i=1}^{6} \Delta_i^\Gamma(1) = 2 \]
- **\( u_2(\Gamma) \):** 
  \[ u_2(\Gamma) = \frac{1}{6} \sum_{i=1}^{6} \Delta_i^\Gamma(2) = 5 \]
- **\( u_3(\Gamma) \):** 
  \[ u_3(\Gamma) = \frac{1}{6} \sum_{i=1}^{6} \Delta_i^\Gamma(3) = 35 \]
Stability of Coalitions: Core

Consider a characteristic function game \( \Gamma = \{ \mathcal{N}, v \} \) with an outcome \((\mathcal{C}, u)\).

Let \( u(C) \) denote the total payoff of a coalition \( C \) under \( u \).

Given a coalition \( C \), if \( u(C) < v(C) \), some agents can abandon \( C \) and form their own coalition.

**Definition**

A utility profile \( u \) is **stable** through a coalition \( C \) if

\[
\sum_{i \in C} u_i \geq v(C).
\]

**Definition**

*Core* is defined as the set of all stable utility profiles, which is denoted as

\[
\mathbb{C} = \left\{ u \in \mathbb{R}_+^N \left| \sum_{i \in C} u_i \geq v(C), \text{ for all } C \subset \mathcal{N} \right. \right\}.
\]
Core: An Example

Consider a characteristic function game \( \Gamma = \{\mathcal{N}, v\} \), where \( \mathcal{N} = \{1, 2, 3\} \) and

\[
v(C) = \begin{cases} 
1 & \text{if } C = \{1\} \\
0 & \text{if } C = \{2\} \\
1 & \text{if } C = \{3\} \\
4 & \text{if } C = \{1, 2\} \\
3 & \text{if } C = \{1, 3\} \\
5 & \text{if } C = \{2, 3\} \\
8 & \text{if } C = \{1, 2, 3\}.
\end{cases}
\]

\[\begin{align*}
\text{Then, the utility profiles are those such that } & u_1 + u_2 + u_3 = 8 \text{ such that } u_1 \geq 1, u_2 \geq 0 \text{ and } u_3 \geq 1. \\
\text{This is a hyperplane with vertices } & (7, 0, 1), (1, 0, 7), \text{ and } (1, 6, 1).
\end{align*}\]

**Is core always non-empty?**
Core in Convex and Simple Games

**Theorem**

Any convex game $\Gamma = (\mathcal{N}, v)$ has a non-empty core.

**Definition**

In a characteristic function game $\Gamma = (\mathcal{N}, v)$, a player $i$ is called a **veto player**, if $v(C) = 0$ for all $C \subseteq \mathcal{N} - \{i\}$.

**Theorem**

A simple game $\Gamma = (\mathcal{N}, v)$ has a non-empty core, if and only if there is a veto player in $\mathcal{N}$. Moreover, a utility profile $u$ is in the core of $\Gamma$ if and only if $u_i = 0$ for every player $i$, who is not a veto player in $\Gamma$. 

Core and Superadditive Covers

**Definition**

\( \Gamma^* = (\mathcal{N}, v^*) \) is called a *superadditive cover* of \( \Gamma = (\mathcal{N}, v) \) if, for every coalition \( C \subseteq \mathcal{N} \),

\[
v^*(C) = \max_{\mathcal{P}_C} \sum_{C_i \in \mathcal{P}_C} v(C_i),
\]

where \( \mathcal{P}_C \) denotes a partition of the coalition \( C \).

Consider \( \Gamma = (\mathcal{N}, v) \): \( \mathcal{N} = \{1, 2, 3\} \) and

\[
v(C) = \begin{cases} 
5 & \text{if } C = \{1\} \\
0 & \text{if } C = \{2\} \\
0 & \text{if } C = \{3\} \\
1 & \text{if } C = \{1, 2\} \\
1 & \text{if } C = \{1, 3\} \\
1 & \text{if } C = \{2, 3\} \\
1 & \text{if } C = \{1, 2, 3\}.
\end{cases}
\]

Its superadditive cover \( \Gamma^* = (\mathcal{N}, v^*) \) is

\[
v^*(C) = \begin{cases} 
5 & \text{if } C = \{1\} \\
0 & \text{if } C = \{2\} \\
0 & \text{if } C = \{3\} \\
5 & \text{if } C = \{1, 2\} \\
5 & \text{if } C = \{1, 3\} \\
1 & \text{if } C = \{2, 3\} \\
6 & \text{if } C = \{1, 2, 3\}.
\end{cases}
\]

**Theorem**

A characteristic function game \( \Gamma = (\mathcal{N}, v) \) has a non-empty core if and only if its superadditive cover \( \Gamma^* = (\mathcal{N}, v^*) \) has a non-empty core.
Summary

- **Characteristic function game**: How to model players' rationality in coalitional games?
- **Subclasses**: Are there any special games that result in some specific coalitions?
- **Shapley value**: How to distribute a coalition’s value in a fair manner amongst its members?
- **Core**: What is a stable coalition?